Markovian optical modes

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We describe the transition of a set of optical modes following a Markov chain process, where the mean value of the amplitude converge to a new type of partially coherent mode, with the property that the coherence features are easily tunable with the parameters of the chain. The amplitude of the resulting mode depends on the probability transition of the chain. As a prototype, we establish an analogy with gambler's chain ruin, using as a basis for the vector space the Bessel modes of integer order. Computer simulations are shown.

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Mode theory is a powerful model used to study the contemporary optics [1–5]. Following Durnin's definition [6], an optical mode is an exact solution to the Helmholtz equation, $\nabla^2\phi(x,y,z)+K^2\phi(x,y,z)=0$, with the form,

$$\phi(x,y,z)=f(x,y)\exp(i\beta z), \tag{1}$$

where the function $f(x,y)$ satisfies the eigenvalue equation,

$$\nabla_\perp^2 f(x,y)+K^2 f(x,y)=\beta^2 f(x,y). \tag{2}$$

A great variety of modes have been identified solving the corresponding eigenvalue equation in different reference systems [7–11]. The objective of this present Letter is to describe the synthesis of partially coherent modes by means of a stochastic process of the Markov chain type [12,13]. This generates modes with tunable coherence parameters, thereby increasing the potential applications of mode theory. We begin our analysis with the identification of the matrix structure of completely coherent modes, and then use this representation to match it with the transition matrix of a Markov chain. The study is supported by the fact that the set of optical modes propagating in the same direction and with the same phase value β has the structure of a vector space. Without loss of generality, we consider propagation along the z-coordinate, using the set of Bessel modes of integer order as vector space basis,

$$\{e^{i\beta z}J_n(2\pi rd)e^{\mathrm{i}n\theta}\} \quad n=0,\pm1,\pm2,\ldots. \tag{3}$$

This basis is easily identified by the fact that a completely coherent optical mode, also known as diffraction free beam, has an associated frequency representation given by

$$T(u,v)=H(u,v)\delta(u^2+v^2-d^2), \tag{4}$$

where u, v are the spatial frequencies, $H(u,v)$ is an arbitrary function, and δ is the Dirac delta function with circular geometry. The amplitude distribution in the optical field can be obtained using the angular spectrum model, whose amplitude values in cylindrical coordinates, at an arbitrary point p, is given by [3]

$$\varphi(r,\theta,z)=e^{i\beta z}\sum_n J_n(2\pi rd)e^{\mathrm{i}n\theta}\int H(\phi)e^{-\mathrm{i}n\phi}\mathrm{d}\phi. \tag{5}$$

Equation (5) is the general expression of any given amplitude function for a completely coherent mode, and it is a linear combination of Bessel modes. The coefficients for the linear combination are obtained from the modulation function $H(\phi)$ given by

$$\alpha_n=\int H(\phi)e^{-\mathrm{i}n\phi}\mathrm{d}\phi. \tag{6}$$

The geometry and certain key physical features of the mode can be obtained by calculating the mode irradiance distribution, which is obtained by taking the square modulus of the amplitude function,

$$\begin{aligned} I(p)&=|\varphi(r,\theta,z)|^2\\ &=\sum_{n,m}J_n(2\pi rd)e^{i\theta n}\alpha_{nm}\sum_{n,m}J_m(2\pi rd)e^{-i\theta m}, \end{aligned} \tag{7}$$

where

$$\alpha_{nm}=\alpha_n\alpha_m^*=\iint H(\phi)H^*(\phi')e^{\mathrm{i}n\phi}e^{-\mathrm{i}m\phi'}\mathrm{d}\phi\mathrm{d}\phi'. \tag{8}$$

The mode irradiance given in Eq. (7) does not depend on the z-coordinate. This implies morphological invariance along the z-coordinate, thus corroborating the nondiffractive behavior of the optical mode. Rewriting Eq. (7) in matrix form gives

$$I(r) = (J_0(2\pi rd), J_1(2\pi rd)e^{i\theta}, \ldots, J_n(2\pi rd)e^{\mathrm{i}n\theta}\ldots) \begin{pmatrix} |\alpha_{00}|^2 & \alpha_0\alpha_1^* & \alpha_0\alpha_n^* \\ \alpha_1\alpha_0^* & |\alpha_{11}|^2 & \alpha_1\alpha_n^* \\ \alpha_n\alpha_0^* & \alpha_n\alpha_1^* & |\alpha_{nn}|^2 \end{pmatrix} \begin{pmatrix} J_0(2\pi rd) \\ J_1(2\pi rd)e^{-i\theta} \\ J_n(2\pi rd)e^{-\mathrm{i}n\theta} \end{pmatrix}. \tag{9}$$

The central matrix constitutes the fundamental structure on which Markovian modes will be constructed. Each term in the matrix describes the interaction strength between the elements of the basis; for example, the term α_{01} determinates the fraction of energy that can be transferred from the zero-order Bessel mode to the one-order Bessel mode. From Eq. (9), we can identify an important case: all the elements in the interaction matrix are zero except for a single term located on the principal diagonal. When this occurs, we know that the optical field corresponds to a pure mode,

$$\varphi(r,\theta,z) = \alpha_n e^{i\beta z} J_n(2\pi rd)e^{\mathrm{i}n\theta}. \tag{10}$$

This concept will be used below to define the purity of the Markovian mode.

Up to this point, our analysis has been related to completely coherent modes. The partial coherence effects appear when the modulation function is time dependent, i.e., $H(\phi,t)$. Then, each term in the interaction matrix must be replaced by its expected value,

$$J(a) = \begin{pmatrix} \langle|\alpha_0|^2\rangle & \langle\alpha_0\alpha_1^*\rangle & \langle\alpha_0\alpha_n^*\rangle \\ \langle\alpha_1\alpha_0^*\rangle & |\alpha_1|^2 & \langle\alpha_1\alpha_n^*\rangle \\ \langle\alpha_n\alpha_0^*\rangle & \langle\alpha_n\alpha_1^*\rangle & \langle|\alpha_n|^2\rangle \end{pmatrix}, \tag{11}$$

this expression has the form of a coherence matrix.

Considering an ergodic behavior, the coherence matrix elements can be calculated using the angular correlation function given by

$$\langle\alpha_n\alpha_m^*\rangle = \iint \phi\phi' e^{\mathrm{i}n\phi}e^{-\mathrm{i}m\phi'}\rho(\phi,\phi')\mathrm{d}\phi\mathrm{d}\phi', \tag{12}$$

where $\rho(\phi,\phi')$ is the joint probability density function.

Analysing the coherence matrix by files, we can determine how energy is redistributed between all elements of the basis. For example, the k-file determines the energy of the k-order Bessel mode that is interchanged between the other Bessel modes, i.e., all files include information on irradiance distribution from all corresponding elements of the basis.

From the coherence matrix, an accurate quantification of how the irradiance of the i-th Bessel mode is interchanged between the other elements of the basis is obtained by means of file entropy

$$S_i = -\sum_{k=1}^{n}|\alpha_{ik}|\ln\left(\sum_{k=1}^{n}|\alpha_{ik}|\right). \tag{13}$$

The reason for proposing these measurements arises from the fact that entropy allows the establishment of an order relation in the system. In this sense, file entropy is a way to describe strength irradiance between all elements of the basis. From this definition, it is easy to demonstrate that entropy for a pure mode is zero, which means that no interaction with other members of the basis exists. The file entropy values generate a row vector of the form

$$\vec{S} = (S_0 \quad S_1 \quad \ldots \quad S_n)^T, \tag{14}$$

and order relation is established by ordering the entropy values in decreasing order $S_k > S_q > S_p = S_s > \ldots$. This means that the k order Bessel mode generates the maximum irradiance interaction among other members of the basis, the following element in order of importance is the Bessel mode of order q, and that p and s modes contributes in the same proportion to the synthesis of the partially coherent mode.

As an example to illustrate the latter, we obtain the coherence matrix when the joint probability density function is uniform, i.e., $\rho(\varphi,\varphi') = a$. For the calculation of the matrix elements, we use the Eq. (12) obtaining

$$\begin{pmatrix} \pi^2 & \pi & \frac{\pi}{2} & \frac{\pi}{3} & \cdots & \frac{\pi}{n} \\ \pi & 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{\pi}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \cdots & \frac{1}{2n} \\ \frac{\pi}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{9} & \cdots & \frac{1}{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\pi}{n} & \frac{1}{n} & \frac{1}{2n} & \frac{1}{3n} & \cdots & \frac{1}{n^2} \end{pmatrix}. \tag{15}$$

If we consider only the first five modes, the row entropy to the partially coherent mode is

$$\vec{S} = (0.35 \quad 0.21 \quad 0.136 \quad 0.102 \quad 0.083)^T. \tag{16}$$

This means that the pure mode that best fits the partially coherent mode is the zero-order Bessel mode. Finally, to quantify the similarity between the partially coherent mode and the pure mode, we define the vector purity, which is

$$\vec{\sigma} = (1 - S_0' \quad 1 - S_1' \quad \ldots \quad 1 - S_n')^T, \tag{17}$$

where S' is Von Newman entropy [14], calculated using only the elements on the principal diagonal,

$$S_n' = -\frac{\langle|a_n|^2\rangle}{trJ}\ln\left(\frac{\langle|a_n|^2\rangle}{trJ}\right), \tag{18}$$

where trJ is the trace of the coherence matrix. We remark that file entropy is complementary to the analysis presented by Barakhat [14]. Up to this point, the analysis presented here describes the structure of the partially coherent mode, where the ergodic hypothesis is implicit, a sufficient condition when the coherence parameters have a fixed value. The ergodic hypothesis will be eliminated for the synthesis of Markovian modes.

In this section we go a step further in the synthesis of the partially coherent modes. In particular, we demonstrate the synthesis of modes whose coherence parameters are tunable using a Markov chain process. The prototype mode is constructed in analogy with gambler's chain ruin with four states; however, its generalization can be implemented to other Markov chains by means of its matrix transition [12]. The Markov chain is built as follows. Let a gambler have an initial asset of either "1" or "2" units. The gambler beats one unit per trial, and the game is over when his assets reach "0" or "3." The Markov chain is sketched in Fig. 1.

The states "0" or "3" are known as absorbent states and "1" and "2" as transient states. To incorporate the optical features, the set of states $\{0, 1, 2, 3\}$ is matched with Bessel modes $\{J_0, J_1, J_2, J_3\}$. The evolution of the chain is determined by the transition probability matrix, given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ p & 0 & q & 0 \\ 0 & p & 0 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

where p is the probability of losing and q is the probability of winning. Evolution of the Markov chain depends on the initial probability vector, whose form is given by (0, α, β, 0) that satisfies $\alpha + \beta = 1$, where α is the probability that the gambler has "1" assets, and β is the probability that he has "2" assets. Applying successively the transition matrix to the initial vector and after N steps, the outcome vector acquires the form $P(0)$, $P(1)$, $P(2)$, $P(3)$, where

$$\begin{aligned} P(0) &= \alpha \sum_0^N \alpha P^{n+1} q^n + \beta P^2 \sum_0^N P^n q^n, \\ P(1) &= \beta q^n P^{n+1} \quad \text{if } n \text{ even}, \\ P(1) &= \alpha q^n p^n \quad \text{if } n \text{ odd}, \\ P(2) &= \alpha q^n P^{n+1} \quad \text{if } n \text{ even}, \\ P(2) &= \beta q^n p^n \quad \text{if } n \text{ odd}, \\ P(3) &= \alpha q^2 \sum_0^N P^n q^n + \beta \sum_0^N P^n q^{n+1}. \end{aligned} \tag{20}$$

$P(0)$ is the probability that the mode reaches the state "0", recovering the structure of the zero-order Bessel mode, and $P(3)$ is the probability that the mode reaches the three order Bessel mode.

For N tending to infinity, the probabilities $P(1)$ and $P(2)$ tend toward zero. This can be corroborated because the sum of probabilities corresponding to the states "0" or "3" is one. The amplitude of the Markovian mode markedly depends on the sequence of the chain. In Tables (1) and (2), we show the possible probabilistic correlation that follows the chain. When the experiment is performed N times, the average number of appearances the absorbent states are shown in the third column.

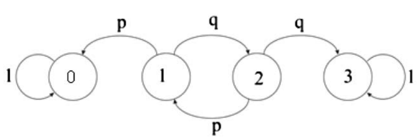

Fig. 1. Diagram of the gambler's chain ruin.

From the values listed in Tables 1 and 2, we can deduce the average number of times that each mode appears. For J_1 and J_2 modes, the average number of times that they appear is $N(\alpha + 1)$ and $N(\beta + 1)$, respectively. Then, the amplitude representation for the Markovian mode is

$$\varphi(r, \theta, z) = A \exp(i\beta z) f(r, \theta) \tag{21}$$

$$f(r,\theta) = N\begin{pmatrix} (\alpha+1)J_1 e^{i\theta} + (\beta+1)J_2 e^{i2\theta} + \\ \{\alpha p + \alpha q p^2 + \beta p^2\}J_0 + \{\beta q + \alpha q^2 + \beta p q^2\}J_3 e^{i3\theta} \end{pmatrix},$$

which is a linear combination containing only the first four elements where the amplitude values depend on both probability transition and initial probability. The corresponding elements of the coherence matrix are

$$\begin{aligned} \langle \alpha_{00} \rangle &= (\alpha p + \alpha q p^2 + \beta p^2)^2, \\ \langle \alpha_{01} \rangle &= \langle \alpha_{10} \rangle (\alpha p + \alpha q p^2 + \beta p^2)(\alpha + 1), \\ \langle \alpha_{02} \rangle &= \langle \alpha_{20} \rangle (\alpha p + \alpha q p^2 + \beta p^2)(\beta + 1), \\ \langle \alpha_{03} \rangle &= \langle \alpha_{30} \rangle (\alpha p + \alpha q p^2 + \beta p^2)(\beta q + \alpha q^2 + \beta p q^2), \\ \langle \alpha_{11} \rangle &= (\alpha + 1)^2, \\ \langle \alpha_{12} \rangle &= \langle \alpha_{21} \rangle = (\alpha + 1)(\beta + 1), \\ \langle \alpha_{13} \rangle &= \langle \alpha_{31} \rangle = (\alpha + 1)(\beta q + \alpha q^2 + \beta p q^2), \\ \langle \alpha_{23} \rangle &= \langle \alpha_{32} \rangle = (\beta + 1)(\beta q + \alpha q^2 + \beta p q^2) \\ \langle \alpha_{33} \rangle &= (\beta q + \alpha q^2 + \beta p q^2)^2. \end{aligned} \tag{22}$$

Table 1. Description of Probability Evolution Starting with State "1"

Initial State "1"		
Sequence of the Chain	Probability	Average No. of Appearances of Absorbent States
$P(1) \to P(0)$	αp	$(N\alpha p)J_0$
$P(1) \to P(2) \to P(3)$	αq^2	$(N\alpha q^2)J_3$
$P(1) \to P(2) \to P(1) \to P(0)$	$\alpha q p^2$	$(N\alpha q p^2)J_0$

Table 2. Description of the Probability Evolution Starting with State "2"

Initial state "2"		
Sequence of the Chain	Probability	Average No. of Appearances of Absorbent States
$P(2) \to P(3)$	βq	$(N\beta q)J_3$
$P(2) \to P(1) \to P(0)$	βp^2	$(N\beta p^2)J_0$
$P(2) \to P(1) \to P(2) \to P(3)$	$\beta p q^2$	$(N\beta p q^2)J_3$

The coherence degree between the elements of the basis is given by the probabilistic correlation divided by the trace of the matrix

$$\gamma_{ij} = \frac{1}{TrJ} \langle \alpha_{ij} \rangle, \tag{23}$$

the file entropy of the Markovian mode is

$$S_i = -\sum_{k=0}^{3} \gamma_{ik} \ln \gamma_{ik}, \tag{24}$$

which depends on the chain parameters. As in the previous section, taking the largest value of file entropy value, we can identify the pure mode that best fits the Markovian mode. The corresponding elements of the purity vector are given by

$$\sigma_i = 1 - S_i', \tag{25}$$

where S' is given by Eq. (18). To clarify these concepts, we performed a computer simulation. In Fig. 2, we show the irradiance convergence for two possible Markovian modes following gambler's chain ruin. The influence of the probabilistic parameters generates an interaction between the constitutive modes generating a tunable interference that is evident in the irradiance distributions. For the simulation, the probabilistic parameters used are shown in the figure caption.

As conclusions, we completely described coherent modes using a matrix representation. With this representation, we identified the parameters that allow the incorporation of partial coherence effects. The study was extended to the synthesis of modes whose coherence degree is tunable according to a Markov chain. We used as prototype the gambler's chain ruin. For description of the coherence features, we proposed the file entropy function, generating a vector entropy, which allows us to deduce an order relation among its components. These results allowed identifying the pure mode that best fits the partial coherence mode. The theory behind this arises from the fact that coherence degree yields information on the correlation between two modes; however, the entropy function includes information on how the irradiance of each mode is interchanged among other modes offering a global description. From the entropy values of the elements on the principal diagonal, we established a purity vector, which is a descriptive measure for the similarity of the partial coherence mode to the pure mode. File entropy is complementary to Von Newman entropy: it offers a deeper understanding of the coherence features of the mode. The theory was constructed using as basis the integer order Bessel modes; however, an arbitrary basis could be incorporated in our treatment. A possible experimental set up can be performed as follows:

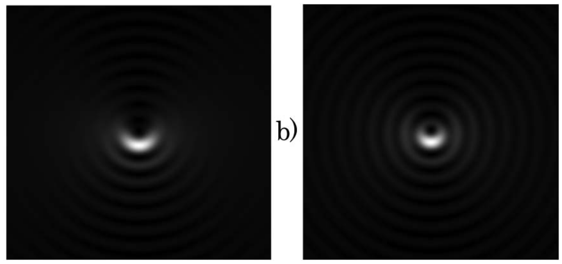

Fig. 2. Computational results for the mean irradiance distribution of the Markovian mode for different values of the probability parameters. The parameters used in (a) are $\alpha = 0.5$, $\beta = 0.5$, $p = 0.5$, $q = 0.5$; in (b) $\alpha = 0.2$, $\beta = 0.8$, $p = 0.2$, $q = 0.8$.

(i) The Fourier transform of Eq. (21) is recorded on a liquid crystal display (LCD).
(ii) The liquid crystal display (LCD) is illuminated with a plane wave.

The Markovian mode is a diffraction-free beam because each compound has the same phase values along the z-coordinate. These kinds of modes offer interesting applications such as tunable spectroscopy, trapping particles, and tunable holography.

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