Journal of Modern Optics

ISSN: 0950-0340 (Print) 1362-3044 (Online) Journal homepage: http://www.tandfonline.com/loi/tmop20

Ion-laser interaction in dispersive regimes: solution using squeeze operators

Arturo Zúñiga-Segundo & Héctor Manuel Moya-Cessa

Journal of Modern Optics, 2015
Vol. 62, No. 17, 1442–1445, http://dx.doi.org/10.1080/09500340.2015.1045436

Ion-laser interaction in dispersive regimes: solution using squeeze operators

Arturo Zúñiga-Segundo[a*] and Héctor Manuel Moya-Cessa[b]

[a] *Departamento de Física, Escuela Superior de Física y Matemáticas, IPN Edificio 9 Unidad Profesional "Adolfo López Mateos", Mexico D.F., Mexico;* [b] *Instituto Nacional de Astrofísica, Óptica y Electrónica, Calle Luis Enrique Erro No. 1, 72840 Sta. María Tonantzintla, Pue., Mexico*

(*Received 2 March 2015; accepted 10 April 2015*)

We study the resonant interaction between a trapped ion and a laser field. By tuning the field intensity far away from the trapping frequency, namely low and high intensity regimes, we arrive at so-called dispersive Hamiltonians that may be simplified via the application of squeezing transformations. We analyse the system dynamics in phase space.

Keywords: ion-laser interaction; squeeze operator; phase space

1. Introduction

By shining light onto a trapped ion it is possible to control its vibrational wavefunction in order to produce some specific quantum states, i.e. engineer non-classical states of the ion's vibrational motion [1–10]. However, such way to control the system is usually done in the low intensity regime, where the Rabi frequency (proportional to the intensity of the laser field) is much smaller than the trapping frequency. Moreover, most of the studies so far can take into account very specific sets of parameters, such as the detuning (the difference between the laser field and the atomic transition frequency) being an integer multiple of the trapping frequency.

Some efforts have been devoted to the solution for some more general sets of parameters [11], namely medium and high intensity regimes. Those regimes are reachable by means of a unitary (similarity) transformation [12] of the full ion-laser Hamiltonian that shows that the ion-laser interaction and the atom-quantized-field interaction are completely equivalent. Then it is possible to apply some known methods to achieve different interactions, and/or to apply some recent methods to achieve a complete analytic solution.

Here, we will focus on two regimes: the high and low intensity ones, that allow adiabatic solutions, i.e. they deliver solvable effective Hamiltonians. We will consider the terms that are usually neglected when arriving to the effective (dispersive) Hamiltonians in order to give a more exact analysis. The appearance of squeeze operator transformations allows exact solutions in this case.

2. Ion-laser Hamiltonian

The ion-laser Hamiltonian in the optical rotating wave approximation (RWA) can be written as

$$\hat{H}_{\text{ion}} = \nu \hat{a}^\dagger \hat{a} + \frac{\delta}{2}\hat{\sigma}_z + \Omega\left(\hat{\sigma}_+ \hat{D}(i\eta) + \hat{\sigma}_- \hat{D}^\dagger(i\eta)\right), \quad (1)$$

where $\hat{D}(i\eta) = \mathrm{e}^{i\eta(\hat{a}+\hat{a}^\dagger)}$ is the Glauber displacement operator, $\delta = \omega_a - \omega_L$ is the laser-ion detuning, ν is the harmonic trapping frequency, ω_a is the atomic transition frequency, ω_L is the field frequency, Ω is the Rabi frequency and η is the Lamb–Dicke parameter.

Based on the unitary transformation

$$\hat{R} = \mathrm{e}^{i\hat{a}^\dagger\hat{a}\frac{\pi}{2}} \mathrm{e}^{\frac{\pi}{4}(\hat{\sigma}_+ - \hat{\sigma}_-)} \mathrm{e}^{-i\frac{\eta}{2}(\hat{a}+\hat{a}^\dagger)\hat{\sigma}_z}, \quad (2)$$

it has shown that the atom-field and ion-laser interactions are in fact exactly equivalent [12]. The above transformation may be written in the 2×2 basis related with the Pauli-spin matrices and with the help of the displacement operator as

$$\hat{R} = \frac{1}{\sqrt{2}} \mathrm{e}^{i\hat{a}^\dagger\hat{a}\frac{\pi}{2}} \begin{pmatrix} \hat{D}^\dagger(\beta) & \hat{D}(\beta) \\ -\hat{D}^\dagger(\beta) & \hat{D}(\beta) \end{pmatrix}, \quad (3)$$

with $\beta = i\eta/2$, such that $\hat{\mathcal{H}}_{\text{ion}} = \hat{R}\hat{H}_{\text{ion}}\hat{R}^\dagger$, gives precisely the atom-field interaction Hamiltonian

$$\hat{\mathcal{H}}_{\text{ion}} = \nu\hat{a}^\dagger\hat{a} + \Omega\hat{\sigma}_z + \frac{\eta\nu}{2}\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right) + \frac{\delta}{2}\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) + \frac{\nu\eta^2}{4}. \quad (4)$$

*Corresponding author. Email: azuniga@esfm.ipn.mx

By considering the on-resonance interaction $\delta = 0$ we can proceed with a set of small rotations [13],[1] to obtain a diagonal effective Hamiltonian via the unitary transformations

$$\hat{U}_1 = e^{\xi_1(\hat{a}^\dagger\hat{\sigma}_+ - \hat{a}\hat{\sigma}_-)}, \quad \hat{U}_2 = e^{\xi_2(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-)}, \tag{5}$$

with the parameters

$$\xi_1 = \frac{\eta\nu}{2}\left(\frac{1}{2\Omega+\nu}\right), \qquad \xi_2 = \frac{\eta\nu}{2}\left(\frac{1}{2\Omega-\nu}\right). \tag{6}$$

Taking into account that $\xi_1, \xi_2 \ll 1$, we may cast the Hamiltonian (4) into the effective Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}}_{\text{eff}} &= \hat{U}_1\hat{U}_2\hat{\mathcal{H}}_{\text{ion}}\hat{U}_2^\dagger\hat{U}_1^\dagger, \\ &\approx \nu\hat{a}^\dagger\hat{a} + \Omega\hat{\sigma}_z + \chi_{\text{ion}}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\hat{\sigma}_z \\ &\quad + g\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right)\hat{\sigma}_z. \end{aligned} \tag{7}$$

where $\chi_{\text{ion}} = 2\eta^2\nu^2\Omega/(4\Omega^2 - \nu^2) = 2g$. We have also used the well-known expansion $e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A}, \hat{B}] + \frac{\xi^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \cdots$ and have kept terms to first-order in ξ_1 and ξ_2, i.e.

$$\begin{aligned} \hat{U}_2\hat{\sigma}_+\hat{U}_2^\dagger &\approx \hat{\sigma}_+ + \xi_2\hat{a}^\dagger\hat{\sigma}_z, \\ \hat{U}_2\hat{\sigma}_-\hat{U}_2^\dagger &\approx \hat{\sigma}_- + \xi_2\hat{a}\hat{\sigma}_z, \\ \hat{U}_2\hat{\sigma}_z\hat{U}_2^\dagger &\approx \hat{\sigma}_z - 2\xi_2\hat{a}\hat{\sigma}_+ - 2\xi_2\hat{a}^\dagger\hat{\sigma}_-, \\ \hat{U}_2\hat{a}\hat{U}_2^\dagger &\approx \hat{a} + \xi_2\hat{\sigma}_-, \\ \hat{U}_2\hat{a}^\dagger\hat{U}_2^\dagger &\approx \hat{a}^\dagger + \xi_2\hat{\sigma}_+, \\ \hat{U}_1\hat{\sigma}_+\hat{U}_1^\dagger &\approx \hat{\sigma}_+ + \xi_1\hat{a}\hat{\sigma}_z, \\ \hat{U}_1\hat{\sigma}_-\hat{U}_1^\dagger &\approx \hat{\sigma}_- + \xi_1\hat{a}^\dagger\hat{\sigma}_z, \\ \hat{U}_1\hat{\sigma}_z\hat{U}_1^\dagger &\approx \hat{\sigma}_z - 2\xi_1\hat{a}^\dagger\hat{\sigma}_+ - 2\xi_1\hat{a}\hat{\sigma}_-, \\ \hat{U}_1\hat{a}\hat{U}_1^\dagger &\approx \hat{a} - \xi_1\hat{\sigma}_+, \\ \hat{U}_1\hat{a}^\dagger\hat{U}_1^\dagger &\approx \hat{a}^\dagger - \xi_1\hat{\sigma}_-. \end{aligned}$$

Note that the Hamiltonian (7) only involves diagonal elements. i.e.

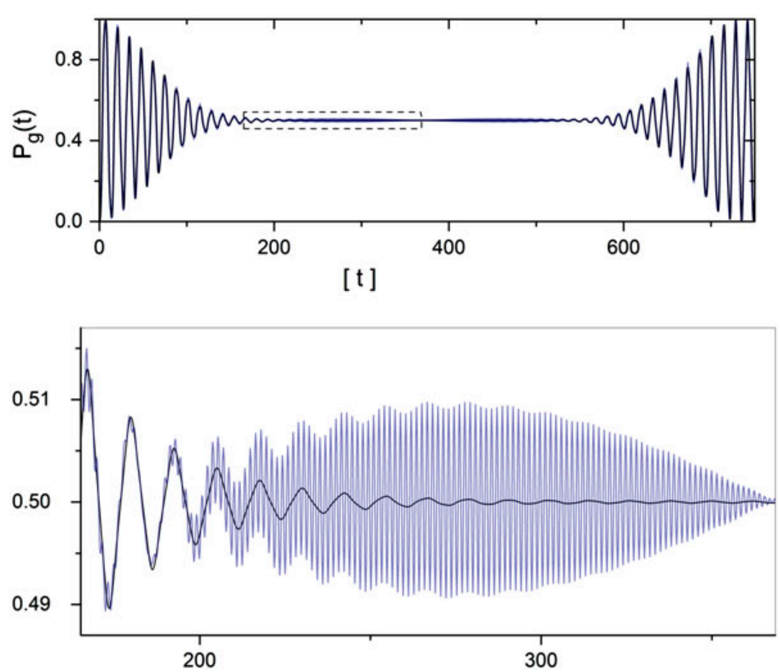

Figure 1. Probability to find the ion in its ground state as a function of time. The parameters used are $\eta = 0.08$, $\Omega = 0.250$, $\nu = 1$ and $\alpha = 3$. The figure below is a larger view of the “steady” region shown in the rectangle above. (The colour version of this figure is included in the online version of the journal.)

$$\hat{\mathcal{H}}_{\text{eff}} = \begin{bmatrix} (\nu+\chi_{\text{ion}})\hat{a}^\dagger\hat{a} + g(\hat{a}^2+\hat{a}^{\dagger 2}) + (\Omega+g) & 0 \\ 0 & (\nu-\chi_{\text{ion}})\hat{a}^\dagger\hat{a} - g(\hat{a}^2+\hat{a}^{\dagger 2}) - (\Omega+g) \end{bmatrix}, \tag{8}$$

on which we perform a squeeze transformation, in order to obtain the transformed Hamiltonian in a simpler form

$$\hat{\mathcal{H}}_\chi = \hat{\mathcal{S}}\hat{\mathcal{H}}_{\text{eff}}\hat{\mathcal{S}}^\dagger = \begin{bmatrix} \omega_e\hat{a}^\dagger\hat{a} + \Omega + \frac{1}{2}(\omega_e-\nu) & 0 \\ 0 & \omega_g\hat{a}^\dagger\hat{a} - \Omega + \frac{1}{2}(\omega_g-\nu) \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{H}}_e & 0 \\ 0 & \hat{\mathcal{H}}_g \end{bmatrix}, \tag{9}$$

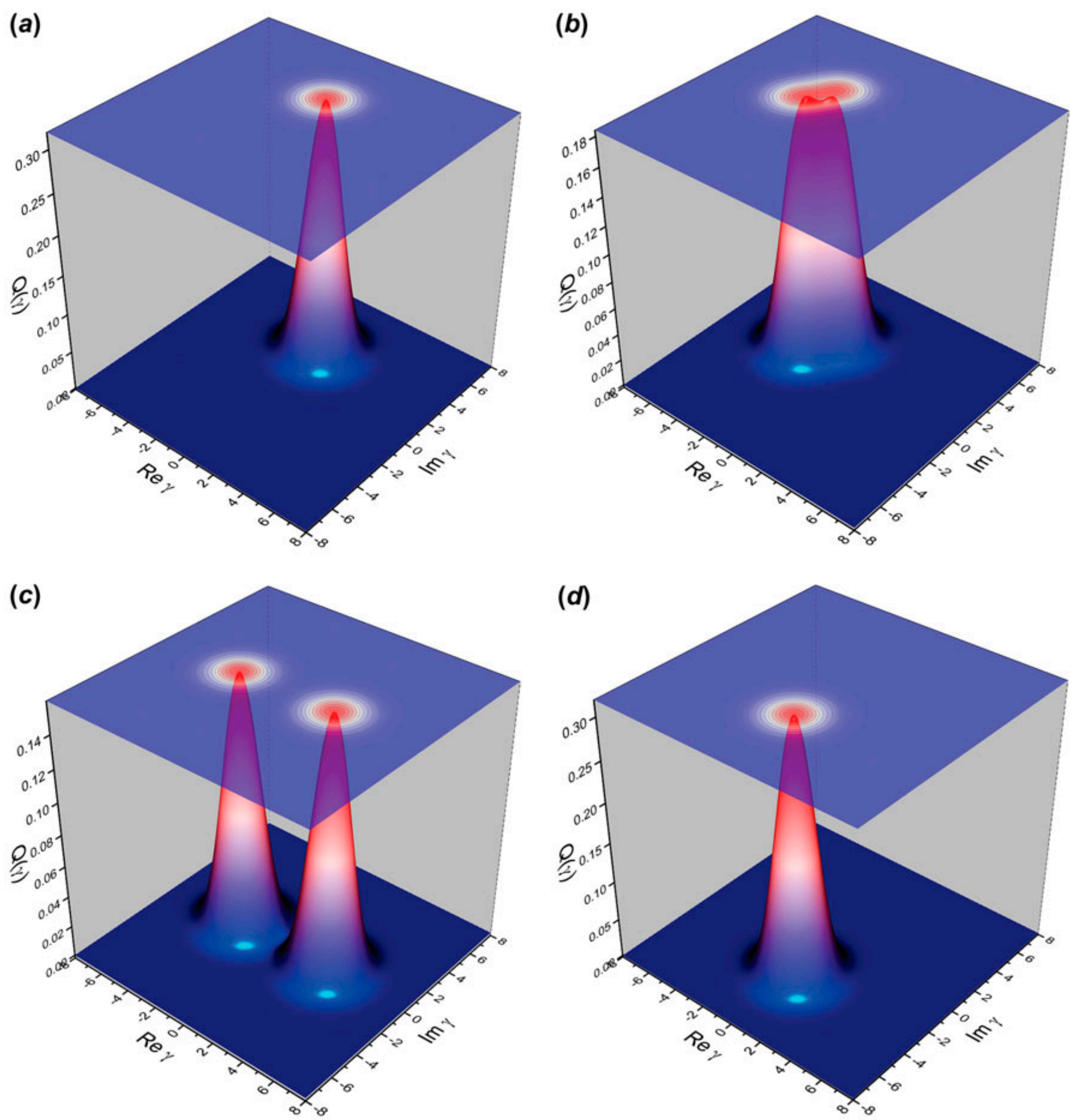

Figure 2. Husimi Q-function. The parameters are the same as in Figure 1 and the times are: (a) $t = 0$, (b) $t = 85.019$, (c) $t = 367.56$, (d) $t = 735.13$. (The colour version of this figure is included in the online version of the journal.)

with

$$\hat{\mathcal{S}} = \begin{bmatrix} \hat{S}(r_e) & 0 \\ 0 & \hat{S}(r_g) \end{bmatrix}, \tag{10}$$

and $\omega_e = \sqrt{\nu^2 + 4g\nu}, \omega_g = \sqrt{\nu^2 - 4g\nu}, r_e = \frac{1}{4}\ln\left[\frac{\nu}{\nu+4g}\right]$ and $r_g = \frac{1}{4}\ln\left[\frac{\nu}{\nu-4g}\right]$.

3. Dynamics and phase space

With the help of the above transformations, the evolution operator $U(t) = \exp(-\mathrm{i}t\hat{H}_{\mathrm{ion}})$ may readily be applied to an initial condition $|\,\psi(0)\rangle$ to render the evolved wave function

$$|\psi(t)\rangle \;= e^{-\mathrm{i}t\hat{H}_{\mathrm{ion}}} \mid \psi(0)\rangle = \hat{R}^{\dagger}\hat{\mathcal{S}}^{\dagger}\mathrm{e}^{-\mathrm{i}t\hat{\mathcal{H}}_{\mathrm{chi}}}\hat{\mathcal{S}}\hat{R} \mid \psi(0)\rangle. \tag{11}$$

This may become tedious given the number of unitary transformations used, but after some algebra we may write the Husimi Q-function for the vibrational wave function as

$$Q(\gamma) = \frac{1}{\pi}\left[|\langle\gamma \mid \psi_e\rangle|^2 + \left|\langle\gamma \mid \psi_g\rangle\right|^2\right], \tag{12}$$

where

$$\begin{aligned}\langle\gamma \mid \psi_e\rangle = \langle\gamma \mid \hat{D}(i\eta/2)\hat{S}(r_e)\mathrm{e}^{-\mathrm{i}t\hat{\mathcal{H}}_e}\hat{S}^{\dagger}(r_e)\hat{D}^{\dagger}(i\eta/2) \mid i\alpha\rangle \\ + \langle\gamma \mid \hat{D}(i\eta/2)\hat{S}(r_g)\mathrm{e}^{-\mathrm{i}t\hat{\mathcal{H}}_g}\hat{S}^{\dagger}(r_g)\hat{D}^{\dagger}(i\eta/2) \mid i\alpha\rangle,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\langle\gamma \mid \psi_g\rangle = \langle\gamma \mid \hat{D}^{\dagger}(i\eta/2)\hat{S}(r_e)\mathrm{e}^{-\mathrm{i}t\hat{\mathcal{H}}_e}\hat{S}^{\dagger}(r_e)\hat{D}^{\dagger}(i\eta/2) \mid i\alpha\rangle \\ - \langle\gamma \mid \hat{D}^{\dagger}(i\eta/2)\hat{S}(r_g)\mathrm{e}^{-\mathrm{i}t\hat{\mathcal{H}}_g}\hat{S}^{\dagger}(r_g)\hat{D}^{\dagger}(i\eta/2) \mid i\alpha\rangle.\end{aligned} \tag{14}$$

In Figure 1, we plot the probability to find the ion in its ground state provided it was initially in the excited state and the initial vibrational wave function was a coherent state with imaginary amplitude, $i\alpha$. In Figure 2, we plot the Husimi Q-function for the same parameters as Figure 1 and different times. The effects produced by the squeezing properties that arise because the terms proportional to the creation and annihilation operators squared in Equation (8) may be seen in the enlarged part of Figure 1, where the collapse region is shown to have little oscillations. The effect, however, is not noticeable in phase space, where the Q-function behaves as expected: splitting into two bumps and its later recombination.

Finally, although the evolution of the initial bump in phase space produced by the initial coherent state splits into two bumps, each bump has different frequencies ω_g and ω_e but neither in counterrotate way on the circle $|\,\alpha\,| = 3$, because we did not transform to a frame rotating at frequency ν. However, the collapse of the oscillations is still due to the splitting into the two bumps, and the revival of oscillations to their recombination [14,15].

4. Conclusions

We have shown that a more exact analysis may be obtained in the “dispersive regimes”. This may be done via a squeeze operator that allows to diagonalize the Hamiltonian (8) in terms of simple number operators that then are easy to handle. Although the application of all the transformations is tedious, the manipulation of the equations is straightforward as the evolution operator may be written in terms of well-known operators which allows to find the dynamics and see small oscillations in the probability of finding the ion in its ground state as a function of time.

Disclosure statement

No potential conflict of interest was reported by the authors.

Note

1. We do not do adiabatic elimination, but instead a set of small rotation as we think both methods of obtaining an effective Hamiltonian give us the same information [13].

References

[1] Cirac, J.I.; Zoller, P. *Phys. Rev. Lett.* **1995,** *74,* 4091–4094.
[2] Jonathan, D.; Plenio, M.B.; Knight, P.L. *Phys. Rev. A* **2000,** *62,* 042307.
[3] Meekhof, D.M.; Monroe, C.; King, B.E.; Itano, W.M.; Wineland, D.J. *Phys. Rev. Lett.* **1996,** *76,* 1796–1799.
[4] Wallentowitz, S.; Vogel, W. *Phys. Rev. A* **1997,** *55,* 4438–4442.
[5] Wallentowitz, S.; Vogel, W.; Knight, P.L. *Phys. Rev. A* **1999,** *59,* 531–538.
[6] Kis, Z.; Vogel, W.; Davidovich, L. *Phys. Rev. A* **2001,** *64,* 033401.
[7] de Matos Filho, R.L.; Vogel, W. *Phys. Rev. Lett.* **1996,** *76,* 608–611.
[8] de Matos Filho, R.L.; Vogel, W. *Phys. Rev. A* **1996,** *54,* 4560–4563.
[9] de Matos Filho, R.L.; Vogel, W. *Phys. Rev. A* **1998,** *58,* R1661–R1664.
[10] Moya-Cessa, H.; Tombesi, P. *Phys. Rev. A* **2000,** *61,* 025401.
[11] Moya-Cessa, H.M.; Soto-Eguibar, F.; Vargas-Martínez, J.M.; Juárez-Amaro, R.; Zúñiga-Segundo, A. *Phys. Rep.* **2012,** *513,* 229–261.
[12] Moya-Cessa, H.; Jonathan, D.; Knight, P.L. *J. Mod. Opt.* **2003,** *50,* 265–273.
[13] Klimov, A.; Sánchez-Soto, L.L. *Phys. Rev. A* **2000,** *61,* 063802.
[14] Eiselt, J.; Risken, H. *Phys. Rev. A* **1991,** *43,* 346–360.
[15] Matsuo, K. *Phys. Rev. A* **1994,** *50,* 649–657.