# Elegant Cartesian Laguerre–Hermite–Gaussian laser cavity modes

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In this work, we present a new family of modes of confocal resonators eigenfunctions of the Fraunhofer diffraction integral, the elegant Cartesian Laguerre–Hermite–Gaussian modes. We show that these modes can be single-pass or round-trip eigenmodes of the resonator depending on the focal distance of the mirrors and their separation. We study their properties and compare them to the well known normal and elegant Hermite and Laguerre–Gauss modes of laser resonators. The new family of modes are not structurally stable on propagation as normal Gaussian modes nor present a monotonic intensity evolution as the normal and elegant Gaussian modes. We also demonstrate that on propagation, they present the self-healing property. © 2015 Optical Society of America

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It is well known that the paraxial wave equation accepts different families of propagating modes being the most known for the normal Hermite–Gauss (HG) modes in Cartesian coordinates and the normal Laguerre–Gauss (LG) modes in cylindrical coordinates. These families are infinite, and on propagation, they are structurally stable maintaining their shape, spreading transversely their energy, and reducing their amplitude. These propagating modes can also be the internal modes of confocal laser resonators [1,2]. About a decade later of HG and LG beams having been introduced, Siegman proposed a variant of the HG modes that he referred to as the elegant form of the Hermite–Gauss (eHG) beams. The main feature is that they show a mathematical symmetry between the argument of the Gaussian function and that of the Hermite functions [3]. These kind of modes were found useful in the description of the refraction and reflection of normal Gaussian beams at dielectric interfaces and also in the propagation of beams emerging from a complex-graded index medium [4,5]. Later, paraxial and nonparaxial elegant Laguerre–Gauss (eLG) beams were formally demonstrated [6–8].

Almost three decades later the unified form for Hermite–Gauss and Laguerre–Gauss beams was proposed and called Hermite–Laguerre–Gaussian beams [9]. It took just a few years for the corresponding elegant Hermite–Laguerre–Gaussian beams to be demonstrated [10]. These, normal and elegant beams are two dimensional, and they are characterized by the continuous modulation of a parameter that represents the transition from a Cartesian Hermite–Gaussian beam into a circular Laguerre–Gaussian beam or vice versa.

In this Letter, we introduce the family of elegant Cartesian Laguerre–Hermite–Gaussian (eCLHG) beams as cavity modes of confocal resonators. They are one-dimensional from which Cartesian 2D beams can be constructed. The eCLHG cavity modes are not structurally stable on propagation as normal HG and LG beams do. However, since the eCLHG beams are eigenfunctions of the Fraunhofer diffraction integral, we demonstrate that they are eigenmodes of confocal laser cavities. We present their properties and compare them with normal and elegant HG and LG beams.

Optical beams can be described with the paraxial wave equation [3], in dimensionless form it can be written as

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - 4i\frac{\partial u}{\partial z} = 0. \tag{1}$$

The equation is normalized in the transverse plane with respect to the intensity beam waist $w_0$. This parameter is defined assuming the intensity of a Gaussian beam of the form $|E(x,y,z=0)|^2 \propto \exp(-r^2/w_0^2)$, that decays to $1/e$ of its maximum amplitude when $r = \sqrt{x^2+y^2} = w_0$. So the transverse normalization is $\xi = x/w_0$, $\eta = y/w_0$ and $\rho = \sqrt{\xi^2+\eta^2}$. Along the longitudinal axis, the normalization is with respect to the diffraction or Rayleigh length $L_D = kw_0^2/2 = \pi w_0^2/\lambda$ with $k = 2\pi/\lambda$ the wavenumber and $\lambda$ the wavelength of the light beam. The longitudinal normalization is then $z' = z/L_D$. In what follows, we will drop the prime to simplify the notation.

Once normalization is performed to the initial condition of the amplitude, we have $u(\xi,\eta,z=0) = \exp[-(\xi^2+\eta^2)/2]$. We observe that the form of this field amplitude also introduces an extra elegance in the description of the propagation of a Gaussian beam. When inserted into Eq. (1) and propagated, a normalized distance of $z = 2$ the intensity decays by a factor of 2 and the beam area, measured at the beam width, is twice of that of the initial condition.

The infinite family of solutions in Cartesian coordinates of Eq. (1) are the Hermite–Gauss beams [2]

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$$\mathrm{HG}_{mn}(\xi,\eta,z)=\frac{E_0}{w(z)}H_m\left(\frac{\xi}{w(z)}\right)H_n\left(\frac{\eta}{w(z)}\right)\times\exp\left[\frac{-\rho^2}{2w^2(z)}-i\frac{\rho^2}{2R(z)}-i(m+n+1)\Phi(z)\right],\quad(2)$$

with $m$ and $n$ defining the order of the mode, and $E_0$ is the field amplitude factor. The corresponding Laguerre–Gauss family of solutions of (1) in cylindrical coordinates with radial symmetry are [2]

$$\mathrm{LG}_{0,n}(\xi,\eta,z)=\frac{E_0}{w(z)}L_n\left(\frac{\rho}{w(z)}\right)\times\exp\left[\frac{-\rho^2}{2w^2(z)}-i\frac{\rho^2}{2R(z)}-i(n+1)\Phi(z)\right],\quad(3)$$

where again $n$ defines the order of the mode, and the radial symmetry implies that there is not azimuthal dependence indicated by the zero sub-index. Both families of beams share the same dependence on the propagation coordinate for the beam width $w(z)=\sqrt{1+z^2/4}$, the transverse phase front $R(z)=(z/2)[1+(2/z)^2]$, and the Gouy phase shift $\Phi(z)=\tan^{-1}(z/2)$. From these relations, notice that the introduced normalization and initial condition also result in a nice elegant form for the transverse and longitudinal phases of the Gaussian beam when is propagated to $z=2$.

The field amplitudes at $z=0$ are of the form $\mathrm{HG}_{m,n}(\xi,\eta,0)=H_m(\xi)H_n(\eta)\exp(-\rho^2/2)$ and $\mathrm{LG}_{0,n}(\rho,0)=L_n(\rho)\exp(-\rho^2/2)$. Maintaining the condition of unitary Gaussian width intensity, the corresponding normalized elegant modes are [7]

$$\mathrm{eHG}_{m,n}(\xi,\eta)=H_m\left(\frac{\xi}{\sqrt{2}}\right)H_n\left(\frac{\eta}{\sqrt{2}}\right)\exp\left(-\frac{\rho^2}{2}\right),\quad(4)$$

$$\mathrm{eLG}_{0,n}(\rho)=L_n\left(\frac{\rho^2}{2}\right)\exp\left(-\frac{\rho^2}{2}\right).\quad(5)$$

By combining these equations, collapsing the radial coordinate to the $\xi$ axis (setting $\eta=0$), we introduce a new family of one-dimensional elegant Cartesian Laguerre–Hermite–Gauss (eCLHG) beams that at their waist can be written as

$$E_n(\xi)=L_{[n/2]}\left(\frac{\xi^2}{2}\right)H_n\left(\frac{\xi}{\sqrt{2}}\right)\exp\left(-\frac{\xi^2}{2}\right),\quad(6)$$

where $[a]$ is the integer part of $a$. Notice the Laguerre function is no longer defined in a two-dimensional domain but in the one-dimensional axis $\xi$ by collapsing the $\eta$ coordinate. In this sense, the proposed eCLHG modes are different from those reported as generalized Hermite–Laguerre–Gaussian and elegant Hermite–Laguerre–Gauss beams defined in two transverse dimensions [9,10].

These new eCLHG modes, unlike the eHG and eLG modes, are eigenfunctions of the Fourier transform operator, namely

$$E_m(k_\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}E_m(\xi)e^{-ik_\xi\xi}\mathrm{d}\xi,\quad(7)$$

where $k_\xi$ is the spatial frequency [11, 12]. Recalling that this eigenfunction property is used for establishing the confocal Cartesian and cylindrical laser resonator modes, we conclude that the eCLHG beams must also be modes of confocal cavities as the HG and LG beams in Eqs. (2) and (3), respectively [2]. Interestingly enough, we remark that eventhough the eHG and eLG beams in Eqs. (4) and (5) do not satisfy the corresponding two dimensional equation related to Eq. (7), they are still asymmetric modes of a particular confocal resonators as we will see below.

By the separability of the Gaussian function, the modes of two-dimensional resonators can be easily constructed in the next form

$$E_{m,n}(\xi,\eta)=E_m(\xi)E_n(\eta).\quad(8)$$

These 2D beams have a finer rectangular lattice structure and larger extension when compared to LG or HG modes due to the increase on the order of the resulting polynomial in their definition in Eq. (6).

To design the resonator cavity that supports these modes, we use the equivalent lens-waveguide model [13]. The waveguide is composed by a sequence of equal lenses separated a distance of $2f$; each lens described by $l(r)=\exp(-ikr^2/2f)$, being $f$ their focal distance. In normalized units $l(\rho)=\exp(-i\rho^2/\mathcal{F})$ where $\mathcal{F}=f/L_D$ is the normalized focal length of the lens. When an eCLHG beam Eq. (6) passes through a lens of focal distance $\mathcal{F}$ placed at the origin, the field at the focal plane has the same profile but is scaled according to

$$E_{m,n}(\xi,\eta,z=\mathcal{F})=\frac{2}{\mathcal{F}}E_{m,n}\left(\frac{\mathcal{F}}{2}\xi,\frac{\mathcal{F}}{2}\eta,z=0\right).\quad(9)$$

From this expression, we see that in order to have exactly the same field distribution at the mirrors of the cavity, these must be of the same focal length $\mathcal{F}=2$ and be separated by a distance $\mathcal{L}=\mathcal{F}$. Figure 1 shows a typical propagation on the $\xi z$–plane of all the discussed modes within the corresponding unfolded cavity. At the bottom of Fig. 1, the mode for $m=n=4$ is shown and compared with the corresponding normal and elegant LG and HG modes with $n=m=4$. The first and second rows correspond to the normal LG and HG modes that, analogously to the eCLGH mode, show a symmetrical volume within the cavity. Not so for the elegant modes that, nevertheless, reproduce themselves after a round trip. The apparent disappearing of the energy for the eHG is due to the fact that it diffracts away diagonally from the $\xi z$–plane to form four spots at the second mirror [7]. Notice that after reflection from the second mirror (plane at $z=2$ of the unfolded cavity), the mode is recreated at the first mirror (plane at $z=4$). This seems to be in contradiction with Siegman who said that the

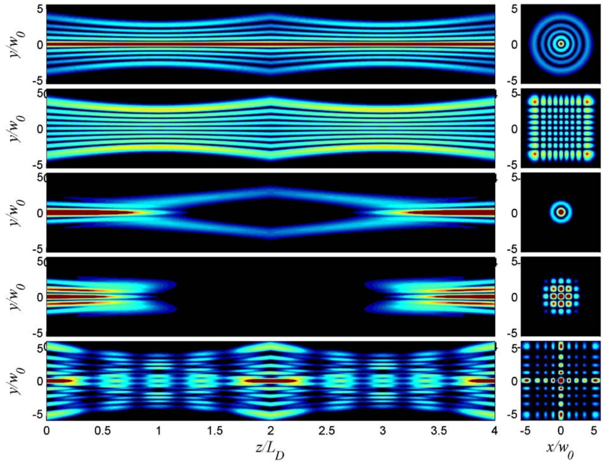


Fig. 1. Unfolded round-trip propagation of confocal cavity modes order $n = 4$ (from top to bottom): normal Laguerre–Gauss, normal Hermite–Gauss, elegant Laguerre–Gauss, elegant Hermite–Gauss, and elegant Cartesian Laguerre–Hermite–Gauss beams. Notice that the order of the eCLHG produces the same number of axial zeros within the cavity.

elegant eigenfunctions are not modes of conventional spherical-mirror optical resonators. We believe that he referred to that they are not symmetrical modes as is the case for the normal HG and LG modes [3].

When $\mathcal{F} \neq 2$ Eq. (9) shows that the field of an eCLHG mode at one mirror is the scaled version of the field at the other, thus there will be modes that repeat only after a complete round trip and will have an asymmetrical volume within the cavity, similar to the elegant HG and LG modes shown in Fig. 1. Thus, the eCLHG modes can be either symmetric as normal Gaussian modes or asymmetric cavity modes as has been proved for elegant Gaussian modes.

As mentioned above, from the bottom image in Fig. 1, it is clear that the eCLHG modes are not structurally stable in the sense that they do not maintain their shape on propagation as normal Hermite–Gauss and Laguerre–Gauss do, their initial maxima interweave creating an

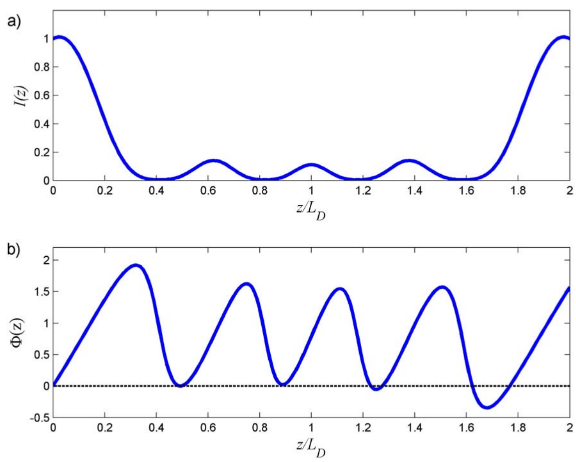


Fig. 2. Single pass intensity and phase on axis for elegant Laguerre–Hermite–Gaussian mode order even with $m = 4$. (a) Within the cavity, the intensity has $n = 4$ minima. The corresponding Gouy phase for LG and elegant modes is also monotonic (not shown).

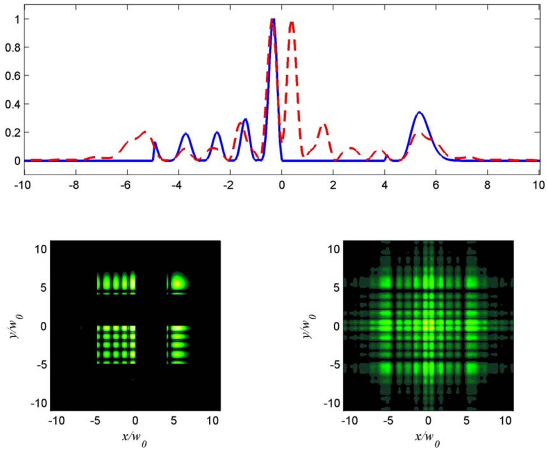


Fig. 3. Self-healing of elegant Laguerre–Hermite–Gaussian mode $m = n = 4$. The far field (dotted red line) has been scaled to make a comparison with the original field intensity (continuous blue line).

oscillating intensity volume within the cavity. We can observe a number of obscured regions that correspond to the order $n$ of the eCLHG cavity mode. There are the same number of zeros in the axial intensity for the eCLHG modes with even $p$, see Fig. 2(a). For the odd modes, there is not similar behavior, and the axial intensity is always zero.

The axial phase of the eCLHG modes has a very striking difference compared to the Gouy phase of any of the normal or elegant modes that has a monotonic evolution. For the eCLHG modes, the axial phase has a sinusoidal-like oscillation with its axis slightly tilted with respect to the propagation axis as can be observed in Fig. 2(b). What is more remarkable is that this tilt can make the axial phase to change its sign. The oscillatory phenomenon occurs, and the tilt is the same for any of the modes irrespective of the order $n$. Also we notice that the number of oscillations of the axial intensity is that of the order mode as it is the number of minima for its axial intensity within the cavity, see Fig. 2.

Finally, one more feature of these modes as eigenfunctions of the Fraunhoffer diffraction integral is that this new family of eCLHG modes can self-heal in the far field when propagating in free space. Figure 3 shows the highly obstructed beam of order $m = n = 4$. The top plot compares the intensities of the initial profile and its scaled far field showing a clear self-healing of the beam with some perturbation due to the diffraction effects of the hard obstruction. The bottom 2D images complement the picture of the self-healing.

In conclusion, we have proposed a new family of modes of confocal resonators. They are eigenfunctions of the Fraunhofer diffraction integral. We have shown that, depending on the focal distance of the mirrors and their separation, they can be single pass or round trip modes. The former fill the cavity symmetrically, while the latter are asymmetrical within the cavity volume. We showed that elegant Hermite and Laguerre–Gaussian modes can be round trip modes of confocal cavities. The eCLHG modes have a very distinctive longitudinal phase behavior when compared to normal and elegant Gaussian modes, it oscillates and can even take negative values. Also, we have shown that this new family of eCLHG modes have the property of self-healing.

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