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Ultracold two-level atom in a quadratic potential

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ABSTRACT

We use a right unitary decomposition to study an ultracold two-level atom interacting with a quantum field. We show that such a right unitary approach simplifies the numerical evolution for arbitrary position-dependent atom–field couplings. In particular, we provide a closed form, analytic time evolution operator for atom–field couplings with quadratic dependence on the position of the atom; e.g. a two-level atom near an extremum of a cavity field mode amplitude. Our approach allows us to show that the center of mass wave function may be squeezed by choosing a proper atom–field initial state.

1. Introduction

The Jaynes–Cummings (JC) model describing the interaction of a two-level atom with a quantized field mode [1] is a solvable working model of the micromaser [2]. In this model, the center of mass velocity of the two-level atom is slow enough to allow controlled atom by atom interaction with the field but fast enough to be described by classical physics; e.g. thermal Rydberg atoms passing through a superconducting cavity showing Rabi oscillations [3]. In the limit case of a two-level atom so slow that its center of mass motion needs to be quantized, the system is described by the following Hamiltonian [4]:

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \omega\hat{a}^\dagger\hat{a} + \frac{\omega_q}{2}\hat{\sigma}_z + g(\hat{z})\left(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right), \tag{1}$$

where the quantized motion of the two-level atom with unitary mass has been taken in the $\hat{z}$-direction with associated canonical momentum $\hat{p}$, the quantum field is described by the annihilation (creation) operators $\hat{a}$ ($\hat{a}^\dagger$) and the frequency ω, and the inner state of the two-level atom by the Pauli matrices $\hat{\sigma}_j$ with $j = z, +, -$ and the transition frequency ω_q. Two regimes of interest can be identified for this model, depending on the ratio between the atomic kinetic energy and the field–atom interaction energy [5]: the intermediate regime, where the mean atomic kinetic energy is of the order of the mean field–atom interaction energy, and the mazer regime, where the kinetic energy is smaller. Amplification via z-motion induced emission of radiation occurs in the latter and gives origin to the mazer name [5–8]. This model is of interest as cavity quantum electrodynamics (cavity-QED) experiments in these two regimes appear feasible with microwave and optical quantum fields [7,9,10]. Also, it is feasible to control or switch off spin interactions of ultracold atoms trapped in optical lattices [11], as well as to address individual sites of such lattices [12,13] at the moment and, in the near future, it may be possible to couple an individual site to a quantum field as cavity-QED has been demonstrated with Bose–Einstein condensates [14,15].

In the theoretical side of the problem, analytic solutions are known for electromagnetic modes described by sinusoidal and mesa functions [5] and sech^2 function [7]. Also, an adiabatic approximation has been proposed by sinusoidal and Gaussian modes [16]. Here, we introduce a right unitary approach to the problem and provide an analytic solution for a quadratic mode. A quadratic mode may be related to an ultracold two-level atom approaching the maximum of a cavity field in an oblique path or trapped in a sinusoidal optical lattice. In the following section, we introduce the right unitary decomposition of the model Hamiltonian for a general quantum field and construct its time evolution operator. Then, we study the resonant case for quadratic couplings and provide a closed form analytic time evolution operator for the system. At this point, we show that an adequate initial state provides us with a squeezed wave function for the center of mass motion of the atom. Finally, we study the interaction of an ultracold excited atom with number and coherent states of the quantum field.

2. Right unitary decomposition

By moving into the frame defined by the excitation number,

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$\hat{a}^\dagger\hat{a} + \hat{\sigma}_z/2$, rotating at frequency ω, we obtain an interaction picture Hamiltonian,

$$\hat{H}_I = \frac{1}{2}\hat{p}^2 + \frac{\delta}{2}\hat{\sigma}_z + g(\hat{z})\left(\hat{a}^\dagger\hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right), \tag{2}$$

where the parameter $\delta = \omega_q - \omega$ is the detuning between the two-level atom and field frequencies. We can follow a right unitary approach [17,18] to decompose this Hamiltonian into the following product:

$$\hat{H}_I = \hat{T}\hat{R}_y\hat{H}_z\hat{R}_y^\dagger\hat{T}^\dagger, \tag{3}$$

where we used a rotation of $\pi/4$ radians around the $\hat{\sigma}_y$ operator,

$$\hat{R}_y = e^{i(\pi/4)\hat{\sigma}_y}, \tag{4}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{5}$$

and the transformation,

$$\hat{T} = \begin{pmatrix} \hat{V} & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{T}^\dagger = \begin{pmatrix} \hat{V}^\dagger & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

The latter is right unitary, $\hat{T}\hat{T}^\dagger = \mathbb{1}$ and $\hat{T}^\dagger\hat{T} \neq \mathbb{1}$, due to the properties of the London exponential of the phase [19,20], also known as Susskind–Glogower [21] operators,

$$\hat{V} = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}, \quad \hat{V}^\dagger = \hat{a}^\dagger\frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}, \tag{7}$$

that yield, in the Fock or number state basis,

$$\hat{V}\hat{V}^\dagger = \mathbb{1}, \tag{8}$$

$$\hat{V}^\dagger\hat{V} = \mathbb{1} - |0\rangle\langle 0|. \tag{9}$$

The new auxiliary Hamiltonian is given by

$$\hat{H}_z = \frac{1}{2}\hat{p}^2 + g(\hat{z})\sqrt{\hat{a}^\dagger\hat{a}}\,\hat{\sigma}_z - \frac{\delta}{2}\hat{\sigma}_x. \tag{10}$$

Typically, a right unitary transformation may act as unitary in just a sector of the corresponding Hilbert space, this is a well known problem in phase operators [22,23]. Here, in order to calculate the evolution operator, $\hat{U}_I(t) = e^{-i\hat{H}_I t}$, it is straightforward to compute each and every term of the corresponding power series to obtain $(\hat{T}\hat{R}_y\hat{H}_z\hat{R}_y^\dagger\hat{T}^\dagger)^j = \hat{T}\hat{R}_y\hat{H}_z^j\hat{R}_y^\dagger\hat{T}^\dagger$ [18]. Thus, the evolution operator of the system is given by the following expression:

$$\hat{U}_I(t) = e^{-i\hat{H}_I t} \tag{11}$$

$$= \hat{T}\hat{R}_y e^{-i\hat{H}_z t}\hat{R}_y^\dagger\hat{T}^\dagger. \tag{12}$$

In other words, the right unitary operators for this Hamiltonian behave like unitary operators in this particular ordering.

In summary, our right-unitary decomposition allows us to construct the time evolution for any given coupling potential for which Eq. (10) is solvable but this does not mean that it is straightforward to interpret the results. In the literature, mazer dynamics for sinusoidal and mesa function [5] and sech^2 [7] are well known. In the following, we will use our approach to solve the quadratic potential mazer and show that it is straightforward to cast the center of mass motion states as squeezed states in this particular case. Furthermore, it seems that a specific operator approach has to be constructed for each and every potential of the form z^j. Thus, the construction of an analytic closed form evolution operator for any given coupling function, $g(\hat{z})$, escapes our efforts at the moment.

3. Time evolution for a quadratic coupling

Here, we will solve the problem for quadratic couplings

$$g_\pm(\hat{z}) = g_0 \pm \frac{|\lambda|}{2}\hat{z}^2. \tag{13}$$

In this case, we can write the auxiliary Hamiltonian in the following form:

$$\hat{H}_{z,\pm} = \begin{pmatrix} \frac{1}{2}\hat{p}^2 + \sqrt{\hat{a}^\dagger\hat{a}}\left(g_0 \pm \frac{\lambda}{2}\hat{z}^2\right) & \delta \\ \delta & \frac{1}{2}\hat{p}^2 - \frac{\lambda}{2}\sqrt{\hat{a}^\dagger\hat{a}}\left(g_0 \pm \frac{\lambda}{2}\hat{z}^2\right) \end{pmatrix}, \tag{14}$$

$$\hat{H}_{z,\pm} = \hat{S}\left[\frac{1}{2}\ln\,\omega(\hat{n})\right]\hat{H}_{0,\pm}\hat{S}^\dagger\left[\frac{1}{2}\ln\,\omega(\hat{n})\right], \tag{15}$$

where the new auxiliary Hamiltonian

$$\hat{H}_{0,\pm} = \begin{pmatrix} \hat{H}_\pm + g_0\sqrt{\hat{n}} & \delta \\ \delta & \hat{H}_\mp - g_0\sqrt{\hat{n}} \end{pmatrix}, \tag{16}$$

contains the standard, $\hat{H}_+ = (\hat{p}^2 + \lambda\hat{z}^2)/2$, and inverted, $\hat{H}_- = (\hat{p}^2 - \lambda\hat{z}^2)/2$, harmonic oscillators, which are equivalent to free propagation and degenerate parametric down-conversion, in that order, or equivalently,

$$\hat{H}_+ = \omega(\hat{n})\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right), \tag{17}$$

$$\hat{H}_- = -\frac{\omega(\hat{n})}{2}\left(\hat{b}^{\dagger 2} + \hat{b}^2\right). \tag{18}$$

Here we defined a frequency in terms of the number operator of the field, $\hat{n} = \hat{a}^\dagger\hat{a}$,

$$\omega\left(\hat{n}\right) = \sqrt{|\lambda|\sqrt{\hat{n}}}, \tag{19}$$

also, we used a boson representation for the atomic center of mass motion

$$\hat{b} = \frac{1}{\sqrt{2}}\left(\hat{z} + i\hat{p}\right), \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}}\left(\hat{z} - i\hat{p}\right), \tag{20}$$

and the action of the squeeze operators,

$$\hat{S}(\hat{\xi}) = e^{-(1/2)\left(\hat{\xi}\hat{b}^{\dagger 2} - \hat{\xi}^\dagger\hat{b}^2\right)}, \tag{21}$$

where the operator $\hat{\xi}$ acts over the cavity field mode, over the position and momentum operators yield

$$\hat{S}(\hat{\xi})\hat{z}\hat{S}^\dagger(\hat{\xi}) = \hat{z}e^{\hat{\xi}}, \quad \hat{S}(\hat{\xi})\hat{p}\hat{S}^\dagger(\hat{\xi}) = \hat{p}e^{-\hat{\xi}}. \tag{22}$$

Note that each and every Fock state of the field, $|k\rangle_f$, defines a bipartite center of mass-field mode, $\left\{|j\rangle_{CM}|k\rangle_f\right\}$ with $j = 0, 1, 2, \ldots$, and auxiliary frequency $\omega(k) = \sqrt{|\lambda|\sqrt{k}}$.

The time evolution operator of such a model is given by

$$\hat{U}_{I,\pm}(t) = \hat{T}\hat{R}_y\hat{S}e^{-i\hat{H}_{0,\pm}t}\hat{S}^\dagger\hat{R}_y^\dagger\hat{T}^\dagger. \tag{23}$$

For the sake of simplicity, let us consider the case of an atom and cavity field on resonance, $\delta=0$. This allows us to construct a closed form evolution operator

$$\hat{U}_{I,\pm}(t) = \hat{T}\hat{R}_y\begin{pmatrix} e^{-ig_0\sqrt{\hat{n}}t}\hat{S}_{\pm}(\hat{n},t) & 0 \\ 0 & e^{ig_0\sqrt{\hat{n}}t}\hat{S}_{\mp}(\hat{n},t)\end{pmatrix}\hat{R}_y^{\dagger}\hat{T}^{\dagger}, \tag{24}$$

with the auxiliary operators

$$\hat{S}_{+}(\hat{n},t) = \hat{S}\left[\frac{1}{2}\ \ln\ \omega(\hat{n})\right]e^{-i\omega\left(\hat{n}\right)\left[\hat{b}^{\dagger}\hat{b}+1/2\right]t}\hat{S}^{\dagger}\left[\frac{1}{2}\ \ln\ \omega(\hat{n})\right], \tag{25}$$

$$\hat{S}_{-}(\hat{n},t) = \hat{S}\left[\frac{1}{2}\ \ln\ \omega(\hat{n})\right]e^{-(i/2)\omega\left(\hat{n}\right)\left[\hat{b}^{\dagger 2}+\hat{b}^{2}\right]t}\hat{S}^{\dagger}\left[\frac{1}{2}\ \ln\ \omega(\hat{n})\right]. \tag{26}$$

4. Squeezing the atomic center of mass motion

Although on resonance the problem looks trivial as one can construct a bare basis so that the Hamiltonian matrix is block diagonal. Such an approach, for some forms of the function $g(\hat{z})$, may miss some underlying physics. Here, we want to show how the operator approach developed in the last section makes obvious the engineering of squeezed center of mass motion states, which is not so obvious in the block diagonal Hamiltonian approach.

Let us start with an ideal initial wave function, where the quantized field is entangled with the inner states of the atom, and the center of mass motion state is in the coherent state $|\beta\rangle$,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}|k\rangle_f \\ -|k+1\rangle_f\end{pmatrix}|\beta\rangle_{CM}. \tag{27}$$

The time evolved wave function yields

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\hat{T}\hat{R}_y\begin{pmatrix}e^{-ig_0\sqrt{\hat{n}}t}\hat{S}_{+}(\hat{n},t)|k+1\rangle_f|\beta\rangle_{CM} \\ 0\end{pmatrix}, \tag{28}$$

where we have chosen the positive sign in Eq. (13). At an interaction time $t_p = \pi/2\omega(k+1)$, the atomic center of mass motion is described by an squeezed state

$$|\psi(t_p)\rangle = \frac{e^{-i\pi/4}e^{-ig_0\sqrt{k+1}t_p}}{\sqrt{2}}\hat{T}\hat{R}_y|e\rangle|k+1\rangle_f|-i\beta,\ \ln\ \omega(k+1)\rangle_{CM}, \tag{29}$$

where we have used the following fact $\hat{S}[\frac{1}{2}\ \ln\ \omega(\hat{n})]e^{-i(\pi/2)\hat{b}^{\dagger}\hat{b}}\hat{S}^{\dagger}[\frac{1}{2}\ \ln\ \omega(\hat{n})] = \hat{S}[\ \ln\ \omega(\hat{n})]$ at time t_p, the addition of a phase component to a coherent state, $e^{-i(\pi/2)\hat{b}^{\dagger}\hat{b}}|\beta\rangle = |-i\beta\rangle$, with a coherent state defined as $|\beta\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}(\beta^n/\sqrt{n!}(|n\rangle$, and the definition of a squeezed coherent state, namely $\hat{S}[\ln\ \omega(k+1)]|-i\beta\rangle = |-i\beta,\ \ln\ \omega(k+1)\rangle$. Note that the remaining two transformations, $\hat{T}$ and $\hat{R}_y$, involve just the inner state of the atom and the quantized field, therefore leaving the atomic center of mass motion wave function untouched.

5. An excited atom interacting with number and coherent states

Let us assume an ultra-slow atom near the maximum of the trapped coherent field inside a cavity. This allows us to approximate the field-two-level atom coupling by a quadratic function on z. In this case, it is possible to describe the coupling as a quadratic potential and the evolution of the system is given by

$$|\psi(t)\rangle = \hat{U}_I(t)|\psi(0)\rangle. \tag{30}$$

In the most general case, we can consider a two-level atom in a superposition of excited and ground states entering the cavity at z_0, with center of mass linear momentum p_0, and consider some general field state in the cavity,

$$|\psi(0)\rangle = \begin{pmatrix}c_e \\ c_g\end{pmatrix}|\phi\rangle_f|\beta\rangle_{CM}, \tag{31}$$

with the coherent parameter defined as $\beta = \frac{1}{\sqrt{2}}\left(z_0 + ip_0\right)$, $|c_e|^2 + |c_g|^2 = 1$.

A practical example is to consider the two-level atom in the excited state, $c_e = 1$ and $c_g = 0$; then, it is straightforward to calculate quantities of interest, such as the mean value of the two-level atomic inversion,

$$\langle\hat{\sigma}_z(t)\rangle = \mathrm{Re}\left[{}_f\langle\phi|_{CM}\langle\beta|e^{2ig_0\sqrt{\hat{n}+1}t}\hat{S}_{\pm}^{\dagger}\left(\hat{n}+1,t\right)\hat{S}_{\mp}\left(\hat{n}+1,t\right)|\beta\rangle_{CM}|\phi\rangle_f\right]. \tag{32}$$

It is also possible to calculate analytic expressions for the mean position, $\langle\hat{z}(t)\rangle$, momentum, $\langle\hat{p}(t)\rangle$, or even the Q-function of the field but they are not as compact as that of the mean atomic inversion. The simplest case for this scenario is given by a field in a Fock state,

$$|\psi(0)\rangle = |e\rangle|n\rangle_f|\beta\rangle_{CM}, \tag{33}$$

with atomic population inversions

$$\langle\hat{\sigma}_z(t)\rangle = \mathrm{Re}\left[e^{2ig_0\sqrt{n+1}t}e^{-|\beta|^2}\sum_{j,k=0}^{\infty}\frac{\beta^{*j}\beta^k}{\sqrt{j!k!}}e^{i\omega(n+1)(j+1/2)t}\times{}_f\langle n|_{CM}\langle j|\hat{S}_{-}\left(\hat{n}+1,t\right)|k\rangle_{CM}|n\rangle_f\right], \tag{34}$$

for the potential $g_{+}(\hat{z})$, and

$$\langle\hat{\sigma}_z(t)\rangle = \mathrm{Re}\left[e^{2ig_0\sqrt{n+1}t}e^{-|\beta|^2}\sum_{j,k=0}^{\infty}\frac{\beta^{*j}\beta^k}{\sqrt{j!k!}}e^{i\omega(n+1)(k+1/2)t}\times{}_f\langle n|_{CM}\langle j|\hat{S}_{-}\left(\hat{n}+1,t\right)|k\rangle_{CM}|n\rangle_f\right], \tag{35}$$

for the potential $g_{-}(\hat{z})$. Note that the term ${}_f\langle n|_{CM}\langle j|\hat{S}_{-}(\hat{n}+1,t)|k\rangle_{CM}|n\rangle_f$ can be calculated exactly and even approximated for large photon numbers by the following [24].

In order to produce an example related to experimental data, let us consider the information from a cavity-cooling scheme presented in [25] where a single ^{85}Rb atom is passed through a high-finesse cavity, $\mathcal{F} = 4.4\times10^5$, that provides a coupling between the cavity TEM$_{00}$ mode and the $5^2S_{1/2}F = 3 \leftrightarrow 5^2P_{3/2}F = 4$ atomic transition with a strength of $g/(2\pi) = 16$ MHz with an interaction length of 9 μm. For our example, we use the value of the coupling strength as our frequency unit, $g_0 = 1$, and set the square potential strength equal to that value, $\lambda = g_0$, for the sake of simplicity; under this assumption a unit of scaled time is 9.9471 ns. We take the field and the atomic transition frequencies on resonance, $\delta = 0$, suppose an ideal square well trap that covers the

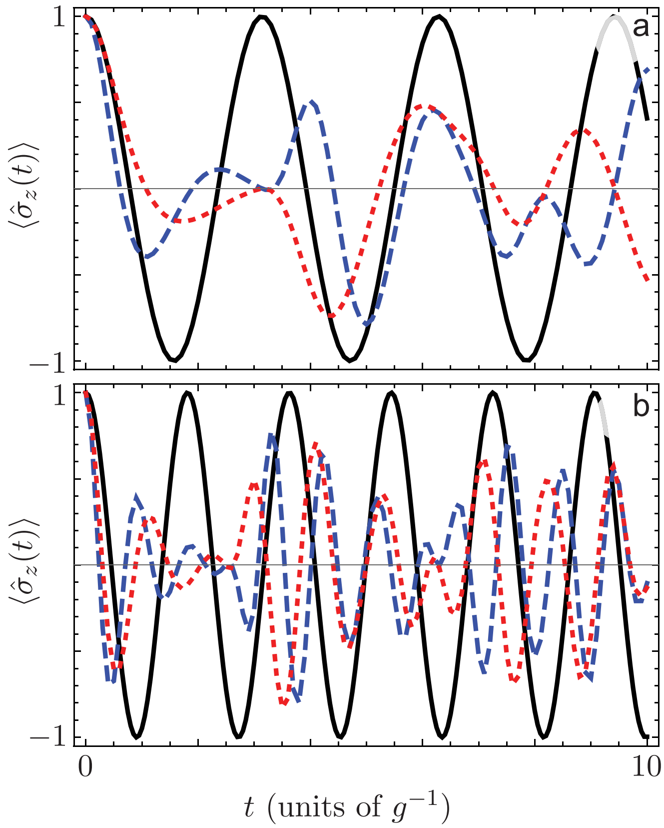

Fig. 1. Time evolution of the population inversion for initial states (a) $|\psi(0)\rangle = |e\rangle|\beta\rangle_{CM}\,|0\rangle_f$ with $\beta = \left(-0.25 + i0.25\right)/\sqrt{2}$ and (b) $|\psi(0)\rangle = |e\rangle|\beta\rangle_{CM}\,|2\rangle_f$ with $\beta = \left(-0.25 + i0.15\right)/\sqrt{2}$ under the potentials $g_+(\hat{z})$ (dashed blue) and $g_-(\hat{z})$ (dotted red) with parameters $g_0 = 1g$ and $\lambda = 1g$ with $g/(2\pi) = 16$ MHz. The Rabi oscillations given by the evolution under Jaynes–Cummings dynamics is also presented (solid black). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

whole z-axis, and artificially place the atom at the initial position $z(0) = 0.6819$ nm that corresponds to an initial value $z(0) = 0.25$ in $\sqrt{\hbar/(\mathrm{mg})}$ units. We assume that the atom has two initial momenta $p(0) = 0.25$ and $p(0) = 0.15$ in units of $\sqrt{\hbar \mathrm{mg}}$ that correspond to temperatures of 23.9937 and 8.6377 mK, in that order. Fig. 1 shows the evolution of the atomic population inversion for the Jaynes–Cummings model and for an ultracold atom under the potentials $g_\pm(\hat{z})$ with the aforementioned parameters, and the atom initially in the excited state, with a coherent center of mass state with coherent parameter $\beta = \left(-0.25 + i0.25\right)/\sqrt{2}$, interacting with a vacuum cavity field, $n=0$, Fig. 1(a), and a slower atom interacting with a two-photon cavity field, $\beta = \left(-0.25 + i0.15\right)/\sqrt{2}$ and $n=2$, Fig. 1(b). Note how the quantization of the atomic center of mass motion induces changes in the dynamics, even in the presence of an empty cavity due to emission and absorption of the initial excitation in the atom. A more realistic scenario involves the atom finding a coherent field in the cavity. Fig. 2 shows the time evolution of the mean population inversion, position and momentum under the potential $g_-(z)$ and the same set of parameters above. Note how the differences in the population inversion are negligible between the initial conditions and how the center of mass movement of the slower atom is trapped before that of the faster atom as expected.

6. Conclusion

We have shown that a right unitary decomposition simplifies the problem of an ultracold two-level atom interacting with a cavity field. In general, it is feasible to use our approach to produce the exact numerical time evolution for arbitrary z-dependent couplings for on- and off-resonance cases. In particular, we show that a quadratic potential can be solved analytically both on- and off-resonance. As an example, we provide a closed form evolution operator on-resonance; here, the evolution operator allows us to calculate closed forms for the mean values of the atomic inversion. The time evolution of the mean intensity of the field, position and momentum of the atomic center of mass can be calculated in closed form but are complicated enough to avoid writing them here. We explored the evolution of an atom originally in the excited state in the presence of number and coherent states.

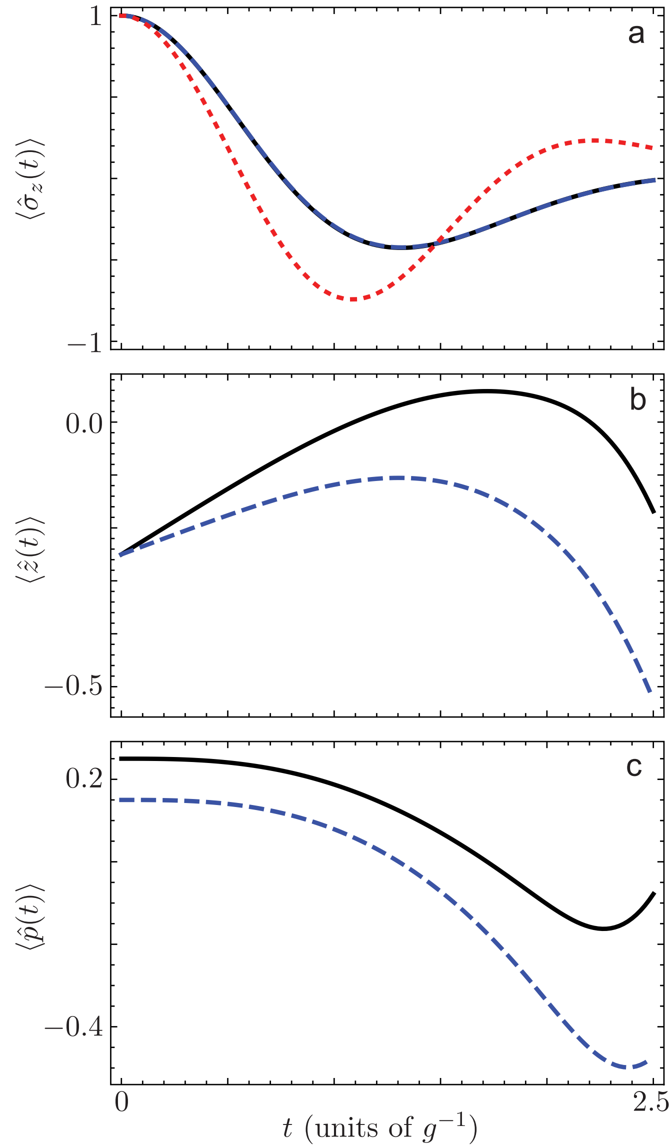

Fig. 2. Time evolution of the mean value of (a) population inversion, (b) position (units of $\sqrt{\hbar/(\mathrm{mg})}$) and (c) momentum (units of $\sqrt{\hbar \mathrm{mg}}$) of the atomic center of mass under the potential $g_-(\hat{z})$ with parameters $g_0 = 1g$ and $\lambda = 1g$. The initial states are $|\psi(0)\rangle = |e\rangle|\beta\rangle_{CM}\,|\alpha\rangle_f$ with coherent parameters $\alpha = 1$ and $\beta = \left(-0.25 + ip_0\right)/\sqrt{2}$ where $p_0 = 0.25$ (solid black) and $p_0 = 0.15$ (dashed blue). In (a) the time evolution of the population inversion under Jaynes–Cummings dynamics (dotted red) is included. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

We want to note that, in theory, it may be possible to use our approach to deal with a generalized potential, via a power series expansion and adequate sets of transformations, but this is unfeasible in practice because the set of transformations for each and every power has to be worked out separately.

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