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The refraction and reflection laws from a complete integral of the eikonal equation and Huygens' principle

Jorge Castro-Ramos[1], Salvador Alejandro Juárez-Reyes[2], Mariana Marcelino-Aranda[3], Paula Ortega-Vidals[2], Gilberto Silva-Ortigoza[2,4], Ramón Silva-Ortigoza[5] and Román Suárez-Xique[2]

[1] Instituto Nacional de Astrofísica, Óptica y Electónica, Apartado Postal 51 y 216, Tonantzintla, Puebla, C. P. 72000, México
[2] Facultad de Ciencias Físico Matemáticas de la Benemérita Universidad Autónoma de Puebla, Apartado Postal 165, 72001, Puebla, Pue., México
[3] Sección de Estudios de Posgrado e Investigación, UPIICSA del Instituto Politécnico Nacional. C.P. 08400, México, DF., México
[4] Departamento de Física, Centro de Investigación y de Estudios Avanzados del I. P. N., Apartado Postal 14-740, 07000 México D.F., México
[5] Áreas de Tecnología de Computación Inteligente & Mecatrónica, CIDETEC del Instituto Politécnico Nacional. Unidad Profesional Adolfo López Mateos, C.P. 07700, México, DF., México

E-mail: gsilva@fcfm.buap.mx

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Abstract

In this work we assume that we have two given optical media with constant refraction indexes, which are separated by an arbitrary refracting surface. In one of the optical media we place a point light source at an arbitrary position. The aim of this work is to use a particular complete integral of the eikonal equation and Huygens' principle to obtain the refraction and reflection laws. We remark that this complete integral associates a new point light source with each light ray that arrives at the refracting surface. This means that by using only this complete integral it is not possible to determine the direction of propagation of the refracted light rays; the direction of propagation is obtained by imposing *two extra conditions* on the complete integral which are equivalent to Huygens' principle (in two dimensions, only one condition is needed). Finally, we establish the connection between the complete integral used here and that derived by using the *k*-function procedure introduced by Stavroudis, which works with plane wavefronts instead of spherical ones.

Keywords: geometric optics, refraction, reflection

(Some figures may appear in colour only in the online journal)

1. Introduction

In this work we assume that we have two given optical media with constant refraction indexes n_1 and n_2, which are separated by an arbitrary refracting surface. In the optical medium with refraction index n_1 we place a point light source at an arbitrary position. From a geometrical optics point of view, the emitted spherical wave is characterized by a family of light rays, or equivalently by a train of spherical wavefronts, such that when they arrive at the refracting surface they experience a change in the direction of propagation. Because of its practical applications [1–8], the main problem in geometrical optics is to compute the refracted light rays, wavefronts, and caustic. Recently we solved this problem [9, 11, 21] by using the procedure of the *k*-function introduced by Stavroudis [2] and generalized by Shealy and

Hoffnagle [12]. In the k-function procedure the refraction law is used explicitly to construct a complete integral of the eikonal equation in the optical medium with refraction index n_2; then, by imposing *two extra conditions* on it, the refracted wavefronts and light rays are computed. The aim of this work is to show that it is not necessary to use the refraction law explicitly because the refraction law is equivalent to the two extra conditions imposed on the complete integral of the eikonal equation. To this end, we use a new complete integral of the eikonal equation in the medium with refraction index n_2, which, with a minor change in the notation, is equivalent to that constructed in [9]. The refraction and reflection laws are obtained by computing the envelope of the wavefronts associated with this complete integral. The envelope conditions are equivalent to Huygens' principle. More precisely, the two extra conditions on the complete integral provide two directions of propagation, one corresponding to the refraction law and the other to the reflection law. Furthermore, we remark on the physical meaning associated with the two complete integrals. The general results are applied to a particular example.

2. The refraction and reflection laws

We assume that the three-dimensional free space is filled out with two different optical media with refraction indexes n_1 and n_2, which are separated by an arbitrary surface locally given by $\mathbf{r}(x, y) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + f(x, y)\hat{\mathbf{z}}$. In the optical medium with refraction index n_1 we place a point light source at $\mathbf{s}$. The optical path length (OPL), Φ, from the point source, $\mathbf{s}$, to an arbitrary point, $\mathbf{X} = X\hat{\mathbf{x}} + Y\hat{\mathbf{y}} + Z\hat{\mathbf{z}}$, in the optical medium with refraction index n_2 is equal to

$$\Phi(\mathbf{X}, x, y) = n_1 |\mathbf{r}(x, y) - \mathbf{s}| + n_2 |\mathbf{X} - \mathbf{r}(x, y)|. \quad (1)$$

Observe that for fixed values of x and y equation (1) is a solution to the eikonal equation in the optical medium with refraction index n_2. This means that $\Phi(\mathbf{X}, x, y)$ is a complete integral of the eikonal equation [13]. To see the physical meaning associated with this complete integral, we compute its associated wavefronts, which are given by all the points $\mathbf{X}$ such that $\Phi(\mathbf{X}, x, y) = C$, that is,

$$|\mathbf{X} - \mathbf{r}(x, y)| = \frac{C}{n_2} - \gamma |\mathbf{r}(x, y) - \mathbf{s}|, \quad (2)$$

where C is a real constant and

$$\gamma \equiv \frac{n_1}{n_2}. \quad (3)$$

Observe that for fixed values of x, y, and C, equation (2) describes a sphere with its center at the point of the refracting surface, $\mathbf{r}(x, y)$, and radius $\frac{C}{n_2} - \gamma |\mathbf{r}(x, y) - \mathbf{s}|$. This means that with each light ray emitted by the point light source at the moment it arrives at the refracting surface it transforms into a new point light source such that its associated spherical wavefronts are given by (2) (see figure 1). In other words, each ray that is emitted by the point light source at the moment it arrives at the refracting surface has the possibility of taking any direction of propagation. Therefore, by using only the information of the complete integral (1) it is not possible to determine the direction of propagation of the refracted light ray. To obtain such a direction we now use Huygens' principle. That is, we assume that the refracted wavefront train is given by the envelope of the wavefronts (2). By definition, the envelope [14, 15] is given by all the points $\mathbf{X}$ that satisfy (2) and the two extra conditions on Φ:

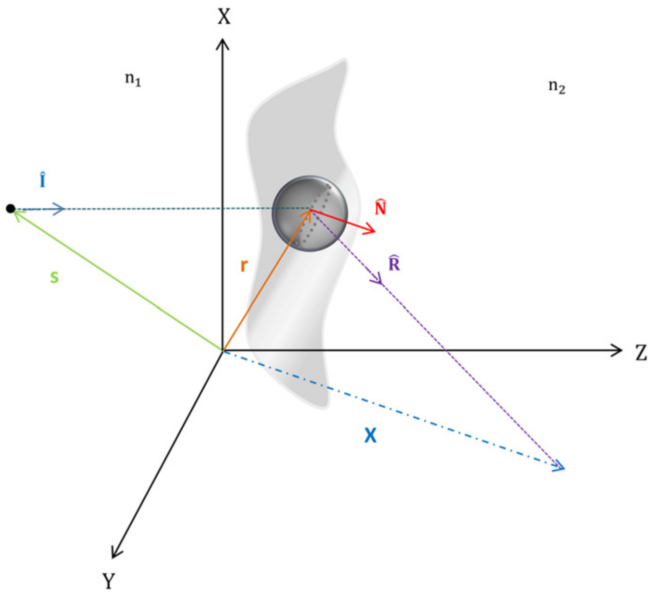

Figure 1. A schematic depiction of two optical media with refraction indexes n_1 and n_2 separated by an arbitrary interface locally given by $z = f(x, y)$. In the optical medium with refraction index n_1 we place a point light source at an arbitrary position $\mathbf{s}$; $\mathbf{r}$ denotes the position of a point on the interface, $\hat{\mathbf{N}}$ denotes the normal vector to the refracting surface, $\hat{\mathbf{I}}$ gives the direction of the incident light ray, and $\hat{\mathbf{R}}$ is the direction of propagation of the refracted light ray, which is determined later. The complete integral (1) associates a new point light source with each light ray that arrives at the refracting surface in such a way that the associated wavefronts of these new point light sources are given by the wavefronts of this complete integral. That is, they are given by equation (2) and are spheres.

$$\Phi_x = -n_2\left(\hat{\mathbf{R}} - \gamma\hat{\mathbf{I}}\right) \cdot \mathbf{r}_x = 0, \quad (4)$$

$$\Phi_y = -n_2\left(\hat{\mathbf{R}} - \gamma\hat{\mathbf{I}}\right) \cdot \mathbf{r}_y = 0, \quad (5)$$

where Φ_x, Φ_y, $\mathbf{r}_x$, and $\mathbf{r}_x$ denote the partial derivatives of Φ and $\mathbf{r}$ with respect to x and y respectively, and

$$\hat{\mathbf{I}} = \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|}, \quad (6)$$

$$\hat{\mathbf{R}} = \frac{\mathbf{X} - \mathbf{r}}{|\mathbf{X} - \mathbf{r}|}. \quad (7)$$

That is, $\hat{\mathbf{I}}$ gives the direction of the incident light ray, and $\hat{\mathbf{R}}$ gives the direction of the refracted light ray. If $\hat{\mathbf{N}}$ denotes the unit vector to the refracting surface then $\hat{\mathbf{N}} \cdot \mathbf{r}_x = 0$ and $\hat{\mathbf{N}} \cdot \mathbf{r}_y = 0$ because $\mathbf{r}_x$ and $\mathbf{r}_y$ are tangent vectors to the refracting surface. Therefore, in general $\mathbf{r}_x$, $\mathbf{r}_y$, and $\hat{\mathbf{N}}$ are linearly independent vectors. This means that any vector in

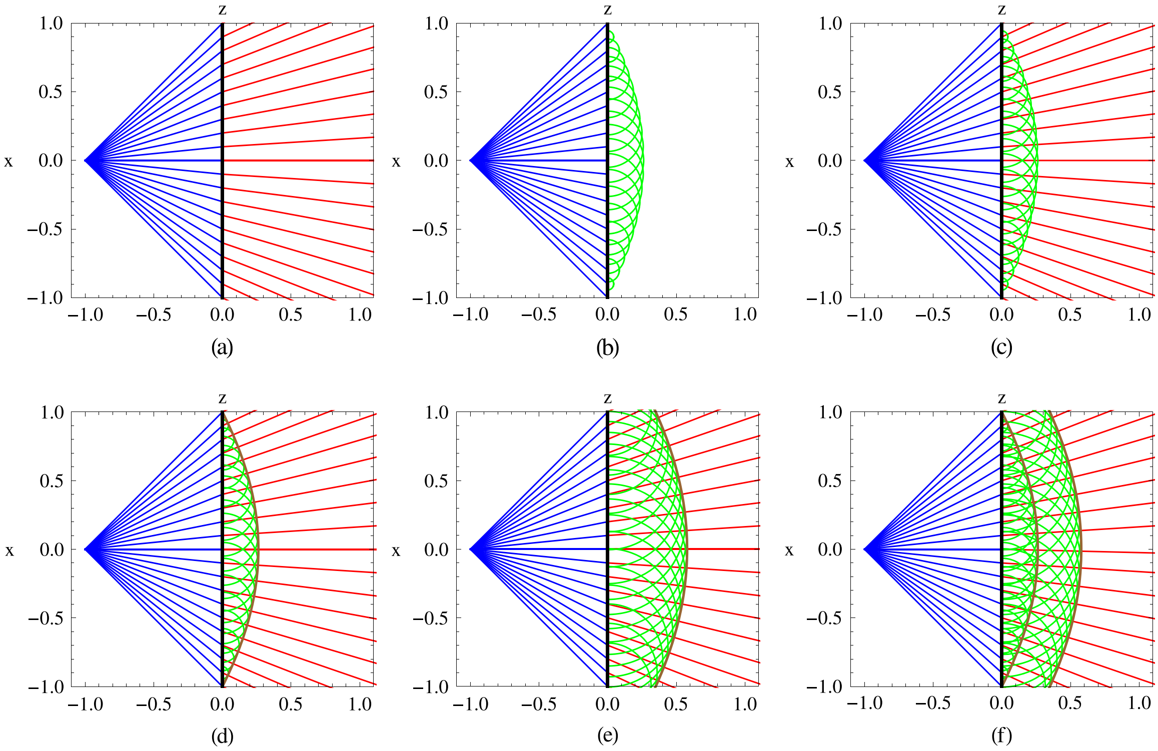

Figure 2. (a) The emitted light rays, the interface, and the refracted light rays. (b) The emitted light rays, the interface, and the part of the spherical wavefronts, given by equation (21) with $C = \sqrt{2}$ cm, that give rise to the refracted wavefront. (c) The superposition of (a) and (b). (d) The superposition of (c) and the envelope or refracted wavefront given by (26) with $C = \sqrt{2}$ cm. (e) As in (d), but now we take $C = (\sqrt{2} + 0.5)$ cm. (f) Superposition of (d) and (e).

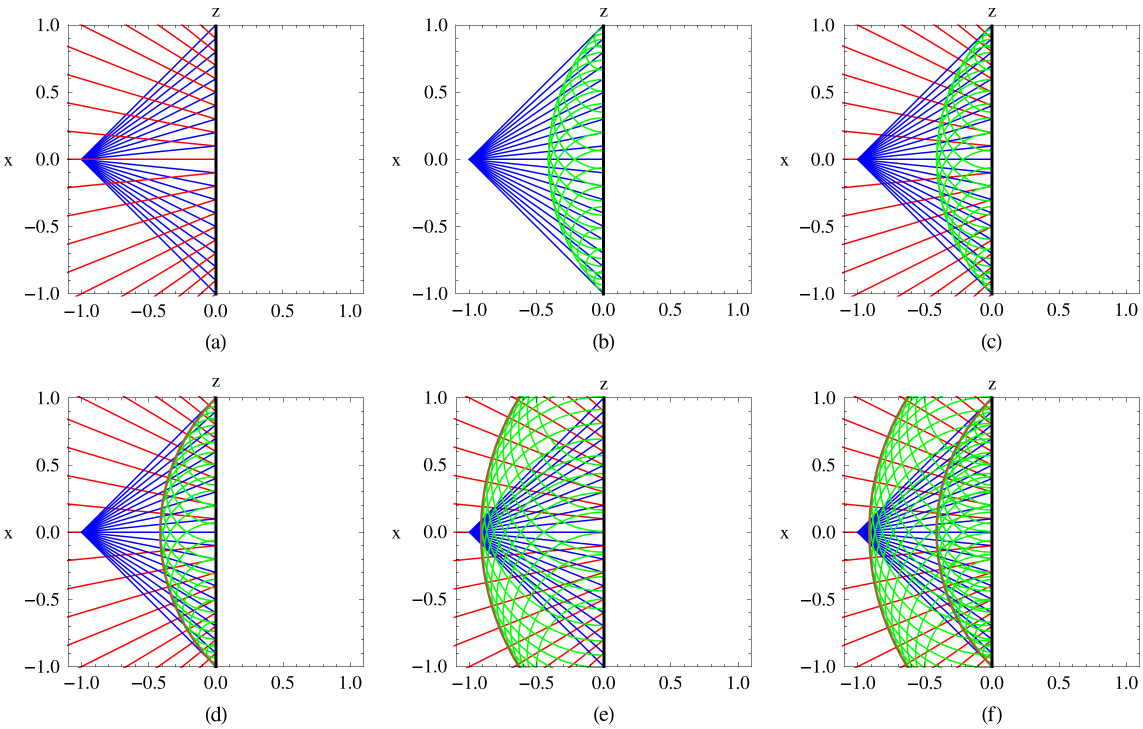

Figure 3. (a) The emitted light rays, the interface, and the reflected light rays. (b) The emitted light rays, the interface, and the part of the spherical wavefronts, given by equation (22) with $C = \sqrt{2}$ cm, that give rise to the reflected wavefront. (c) The superposition of (a) and (b). (d) The superposition of (c) and the envelope or reflected wavefront given by (28) with $C = \sqrt{2}$ cm. (e) As in (d), but now we take $C = (\sqrt{2} + 0.5)$ cm. (f) Superposition of (d) and (e).

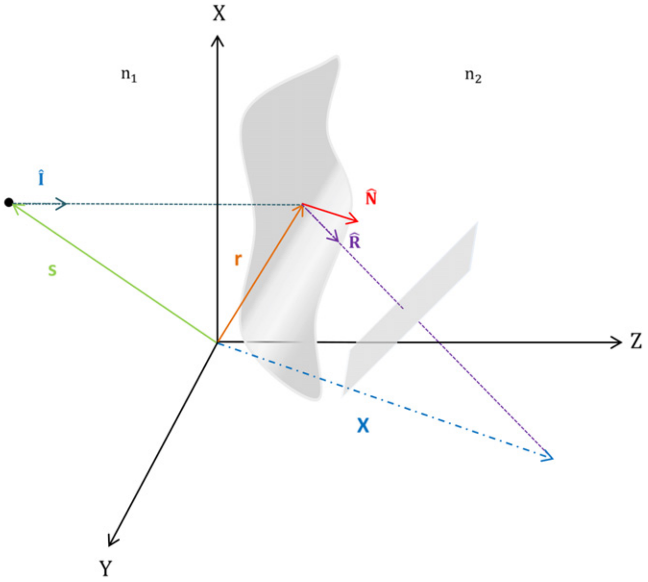

Figure 4. The complete integral (15) associates a new point light source with each light ray that arrives at the refracting surface in such a way that the associated wavefronts of these new point light sources are given by the wavefronts of this complete integral. That is, they are given by $n_2(\mathbf{X}\cdot\hat{\mathbf{R}})+k=C$ and are planes.

the optical medium with refraction index n_2 can be written as a linear combination. From equations (4) and (5) we conclude that

$$\hat{\mathbf{R}}-\gamma\hat{\mathbf{I}}=\Omega\hat{\mathbf{N}}, \tag{8}$$

where Ω is a function determined from the condition $\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}=1$. A direct computation shows that this condition provides two solutions for Ω given by

$$\Omega^{(\pm)}=-\gamma\left(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}}\right)\pm\sqrt{1-\gamma^2\left[1-\left(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}}\right)^2\right]}. \tag{9}$$

Therefore, we conclude that the conditions (4) and (5) on the complete integral (1) of the eikonal equation determine two directions of propagation, which correspond to the refraction and reflection laws in vectorial form.

The refraction law is given by equation (8) with Ω given by equation (9) with the plus sign. That is, by

$$\hat{\mathbf{R}}=\gamma\hat{\mathbf{I}}+\Omega\hat{\mathbf{N}}, \tag{10}$$

$$\Omega\equiv-\gamma\left(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}}\right)+\sqrt{1-\gamma^2\left[1-\left(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}}\right)^2\right]}. \tag{11}$$

Observe that when $n_1=n_2$, then $\gamma=1$, $\Omega=0$, and $\hat{\mathbf{R}}=\hat{\mathbf{I}}$, which is a well-known result.

The reflection law is given by equation (8) with Ω given by equation (9) with the minus sign and taking $n_1=n_2$. That is, by

$$\hat{\mathbf{R}}=\hat{\mathbf{I}}-2(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}})\hat{\mathbf{N}}. \tag{12}$$

For alternative derivations of the refraction and reflection laws see Stavroudis, Tkaczyk, and Bhattacharjee [16–18].

Remember that the envelope of the wavefronts (2) is given by all the points $\mathbf{X}$ that satisfy equations (2), (4), and (5). From the for going results, observe that equations (4) and (5) determine two directions of propagation: one corresponds to the direction of the refracted light rays given by equation (10), and the other corresponds to the direction of the reflected light rays given by equation (12). Therefore, the envelope of the wavefronts (2) has two branches, one corresponding to the refracted wavefronts and the other to the reflected wavefronts. By using equations (2), (7), (10), and (12) a direct computation shows that the refracted wavefronts are given by

$$\mathbf{X}=\mathbf{r}+\left(\frac{C}{n_2}-\gamma\left|\mathbf{r}-\mathbf{s}\right|\right)\left[\gamma\hat{\mathbf{I}}+\Omega\hat{\mathbf{N}}\right], \tag{13}$$

and the reflected wavefronts by

$$\mathbf{X}=\mathbf{r}+\left(\frac{C}{n_1}-\left|\mathbf{r}-\mathbf{s}\right|\right)\left[\hat{\mathbf{I}}-2\left(\hat{\mathbf{I}}\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}}\right]. \tag{14}$$

These results were obtained in the literature [7–9, 12] by using the k-function procedure and the reflection and refraction laws. That is, the complete integral (1) was rewritten in the following form [9]:

$$\Phi(\mathbf{X},x,y)=n_2(\mathbf{X}\cdot\hat{\mathbf{R}})+k, \tag{15}$$

where k is known as the k-function and is given by

$$k(x,y)=n_1\left|\mathbf{r}-\mathbf{s}\right|-n_2(\mathbf{r}\cdot\hat{\mathbf{R}}). \tag{16}$$

Then, depending on the case, $\hat{\mathbf{R}}$ was computed by using explicitly the refraction or reflection laws; and finally, the two extra conditions (4) and (5) were imposed. From the results obtained here we conclude that it is not necessary to use explicitly the refraction or reflection laws to compute the refracted or reflected wavefronts because they are equivalent to the two extra conditions (4) and (5) on the complete integral Φ given by equation (1).

Before closing this section it is important to remark that the envelope of the wavefronts (2) in the optical medium with refraction index n_2 gives the evolution of the refracted wavefronts and is given by equation (13). In the optical medium with refraction index n_1, equation (2) reduces to

$$\left|\mathbf{X}-\mathbf{r}(x,y)\right|=\frac{C}{n_1}-\left|\mathbf{r}(x,y)-\mathbf{s}\right|, \tag{17}$$

and the envelope of these wavefronts is given by equation (14), which gives the evolution of the reflected wavefronts.

3. An example

To clarify the ideas and results presented in this work, we work out explicitly the case when the interface is a plane and the point light source is placed on the z axis. That is,

$$\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}, \tag{18}$$

$$\mathbf{s}=-s\hat{\mathbf{z}}. \tag{19}$$

For this case, the complete integral (1) is given by

$$\Phi(\mathbf{X}, x, y) = n_1\sqrt{x^2 + y^2 + s^2} + n_2\sqrt{(X - x)^2 + (Y - y)^2 + Z^2}, \quad (20)$$

and its associated wavefronts, (2) in the optical medium with refraction index n_2, are given by

$$(X - x)^2 + (Y - y)^2 + Z^2 = \left[\frac{\mathcal{C}}{n_2} - \gamma\sqrt{x^2 + y^2 + s^2}\right]^2, \quad (21)$$

which for fixed values of x, y, and s is a family of spheres with the center at the point $(x, y, 0)$ of the plane interface and radius $\mathcal{C}/n_2 - \gamma\sqrt{x^2 + y^2 + s^2}$. Similarly, its associated wavefronts, (17) in the optical medium with refraction index n_1, are given by

$$(X - x)^2 + (Y - y)^2 + Z^2 = \left[\frac{\mathcal{C}}{n_1} - \sqrt{x^2 + y^2 + s^2}\right]^2, \quad (22)$$

which also are spheres with the center at the points $(x, y, 0)$ and radius $\mathcal{C}/n_1 - \sqrt{x^2 + y^2 + s^2}$. Observe that for given values of $\mathcal{C}$, n_1, and n_2, with n_1, different from n_2, the radii of the spheres (21) and (22) are different because of the difference in the indexes of refraction.

In this example,

$$\mathbf{I} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + s\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + s^2}}, \quad (23)$$

$$\mathbf{N} = \hat{\mathbf{z}}. \quad (24)$$

Using these results in equation (11), a direct computation shows that

$$\Omega = \frac{-\gamma s + \sqrt{\left(1 - \gamma^2\right)\left(x^2 + y^2\right) + s^2}}{\sqrt{x^2 + y^2 + s^2}}. \quad (25)$$

Therefore, using equations (23), (24), and (25) in equation (13) we find that in this case the physical branch of the envelope of the wavefronts (21) is given by

$$\mathbf{X} = \left(1 + \frac{\gamma l}{\sqrt{x^2 + y^2 + s^2}}\right)x\mathbf{x} + \left(1 + \frac{\gamma l}{\sqrt{x^2 + y^2 + s^2}}\right)y\mathbf{y} + \left(\frac{l\sqrt{\left(1 - \gamma^2\right)\left(x^2 + y^2\right) + s^2}}{\sqrt{x^2 + y^2 + s^2}}\right)\mathbf{z}. \quad (26)$$

$$l = \frac{\mathcal{C}}{n_2} - \gamma\sqrt{x^2 + y^2 + s^2} \quad (27)$$

That is, this equation describes the evolution of the refracted wavefronts. In a similar manner, we find that the physical branch of the envelope associated with the wavefronts (22) is given by

$$\mathbf{X} = \left(1 + \frac{\tilde{l}}{\sqrt{x^2 + y^2 + s^2}}\right)x\mathbf{x} + \left(1 + \frac{\tilde{l}}{\sqrt{x^2 + y^2 + s^2}}\right)y\mathbf{y} - \left(\frac{\tilde{l}s}{\sqrt{x^2 + y^2 + s^2}}\right)\mathbf{z}, \quad (28)$$

$$\tilde{l} = \frac{\mathcal{C}}{n_1} - \sqrt{x^2 + y^2 + s^2}. \quad (29)$$

This equation describes the evolution of the reflected wavefronts.

To illustrate the graphic results for this example we consider $s = 1$ cm, $n_1 = 1$, and $n_2 = 1.58$. Furthermore, we present the plots on the plane $y = 0$ because of the symmetry about the z axis. The results are presented in figures 2 and 3 for refraction and reflection respectively.

4. Conclusions

Observe that the complete integrals (1) and (15) are equivalent in the sense that they give the same OPL. The only difference is in their associated wavefronts. As was remarked earlier, the wavefronts associated with (1) are spheres, whereas the wavefronts associated with (15) are given by $n_2(\mathbf{X}\cdot\hat{\mathbf{R}}) + k = \mathcal{C}$ and are planes with normal vector $\hat{\mathbf{R}}$. This means that each ray that is emitted by the point light source in the direction $\hat{\mathbf{I}}$ at the moment it reaches the refracting surface at the point $\mathbf{r}(x, y)$ unfolds into a sphere of light rays; that is, this point is a new point light source and this complete integral associates a train of plane wavefronts with each light ray emitted by the new point light source (see figure 4).

We finish this work with the following remarks:

⋄ To obtain equation (1) we assumed that the point light source is placed at $\mathbf{s}$ and that the observer is placed where the OPL is evaluated, at $\mathbf{X}$. When the optical media are such that the reciprocity principle holds [19], then equation (1) also describes the OPL from a point light source placed at $\mathbf{X}$ to an arbitrary point, $\mathbf{s}$, in the optical medium with refraction index n_1. The wavefronts associated with this new problem are also given by equation (2), but now in that equation $\mathbf{X}$ is fixed and $\mathbf{s}$ is the place where the OPL is evaluated. It is important to point out that this construction has been used to compute the image of an extended object under reflection and refraction [21, 20].

⋄ The results obtained by Shealy and Hoffnagle [12] for a plane wavefront are obtained from equations (13) and (14) by replacing the position of the point light source $\mathbf{s}$ by a vector $\mathbf{r}_0$ that specifies the intersection between the incident light ray and a plane of reference from which the OPL is measured. For example, if the normal vector of the incident plane wavefront is $\hat{z}$ then $\mathbf{s}$ can be replaced by $\mathbf{r}_0 = x\hat{x} + y\hat{y}$. For this particular case, all the relevant objects such as $\hat{\mathbf{R}}$ and Ω have simple expressions (in reference [22] we have worked out all the details for refraction).

⋄ The total internal reflection phenomenon happens when an emitted light ray strikes the refracting surface at an angle larger than a particular critical angle, θ_c, with respect to the normal to the surface. If $n_1 > n_2$ and the incident angle is greater than θ_c, the light ray cannot pass through and is entirely reflected. From equation (11) we see that the critical angle is determined from the following condition:

$$1 - \gamma^2[1 - (\hat{\mathbf{I}} \cdot \hat{\mathbf{N}})^2] = 0, \Leftrightarrow \sin\theta_c = \frac{n_2}{n_1}. \tag{30}$$

A natural and important question is, What happens to the spherical wavefronts (2), (17) and their respective envelope when the total internal reflection phenomenon appears? A partial answer to this question is given in [23]. However, we believe this question deserves to be considered in the future not only for the spherical wavefronts but also for the plane wavefronts associated with the complete integral (15).

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