Intensity-dependent quantum Rabi model: spectrum, supersymmetric partner, and optical simulation

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We study an intensity-dependent quantum Rabi model that can be written in terms of $SU(1,1)$ group elements and is related to the Buck–Sukumar model for the Bargmann parameter $k = 1/2$. The spectrum seems to present avoided crossings for all valid parameter sets and, thus, may be integrable. For a degenerate qubit, the model is soluble, and we construct an unbroken supersymmetric partner for it. We discuss the classical simulation of the general model in photonic lattices and show that it presents quasi-periodic reconstruction for a given initial state and parameter set.

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1. INTRODUCTION

The Jaynes–Cummings model (JCM) [1],

$$\hat{H}_{\mathrm{JC}} = \omega\hat{n} + \frac{\omega_0}{2}\hat{\sigma}_z + g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-), \tag{1}$$

is a theoretical model derived from the minimal coupling [2] between a neutral two-level atom, described by frequency transition ω_0 and Pauli matrices $\hat{\sigma}_j$, and a quantized cavity field mode, described by the frequency ω and the creation (annihilation) operators $\hat{a}^\dagger$ ($\hat{a}$), related to the one-atom maser of cavity-quantum-electrodynamics (cavity-QED) [3,4]; it can also describe the dynamics of a trapped two-level ion in trapped-ion-QED [5] and the coupling of a superconducting qubit interacting with a microwave resonator in circuit-QED [6,7]. The Buck–Sukumar model (BSM) [8], where the coupling between a two-level system and a quantized field depends on the intensity of the field

$$\hat{H}_{\mathrm{BS}} = \omega\hat{n} + \frac{\omega_0}{2}\hat{\sigma}_z + g\left(\hat{a}\sqrt{\hat{n}}\,\hat{\sigma}_+ + \sqrt{\hat{n}}\hat{a}^\dagger\hat{\sigma}_-\right), \tag{2}$$

is a clever theoretical modification of the JCM that leads to a closed-form analytic solution. Its physical realization in the quantum optics laboratory may not be feasible, as it requires a trapped-ion setup driven by a large superposition of field modes [9,10], but it may be classically simulated in arrays of coupled waveguides [11]. Despite a purely theoretical origin, the BSM [8] and its generalization for qubit ensembles [12] have provided analytically tractable models showing periodic decay and revival in the atomic excitation energy [8,12], mean photon number [13], and field squeezing parameters [14] that have attracted the attention of the quantum optics community. It is also well known that the field in the BSM can be described by a $su(1,1)$ algebra [14–16] and that it is possible to interpolate between the JCM and the BSM by choosing a particular q-deformed algebra for the field [17]. The inclusion of the so-called counter-rotating terms obviated by the rotating wave approximation (RWA) into the BSM,

$$\hat{H}_{\mathrm{RBS}} = \omega\hat{n} + \frac{\omega_0}{2}\hat{\sigma}_z + g\left(\hat{a}\sqrt{\hat{n}} + \sqrt{\hat{n}}\hat{a}^\dagger\right)\hat{\sigma}_x, \tag{3}$$

reduces the parameter range where the model is well defined to $g < \omega/2$ due to the underlying $su(1,1)$ symmetry [16,18].

Here we are interested in an intensity-dependent quantum Rabi Hamiltonian that is the simplest generalization of the BSM model without the RWA,

$$\hat{H} = \omega\hat{n} + \frac{\omega_0}{2}\hat{\sigma}_z + g\left(\sqrt{\hat{n}+2k}\hat{a} + \hat{a}^\dagger\sqrt{\hat{n}+2k}\right)\hat{\sigma}_x, \quad k > 0, \tag{4}$$

where a Bargmann parameter value of $k = 1/2$ returns the BSM plus counter-rotating terms. In the following, we will show that this model can be fully written in terms of a $su(1,1)$ algebra due to parity conservation, that it is possible to provide a perturbation theory solution for it in the regime where the qubit transition is negligible, $\omega_0 \ll g$, that a supersymmetric (SUSY) partner can be given for it in this regime, and that both the model and its isospectral partner may be classically simulated by a semi-infinite array of coupled waveguides. In QED, this model may be just a theoretical curiosity; nevertheless, it is an interesting curiosity because it is a solvable model including the counter-rotating terms that are usually neglected. Furthermore, its diagonalization with generalized coherent states of the group SU(1,1), which seems to be valid for just a range of parameters, brings forward what may be an underlying issue in the definition of generalized coherent states [19–21]. In more practical matters, the optical simulation of quantum mechanical models [22,23] is changing the way photonic integrated devices are designed [24–29], and the classical simulation of this model provides a new set of isospectral photonic lattices [30–32].

2. SU(1,1) MODEL AND ITS SPECTRA

The Hamiltonian in Eq. (4) conserves parity, $[\hat{H},\hat{\Pi}]=0$ with $\hat{\Pi}=(-1)^{\hat{n}}\hat{\sigma}_z$. This allows us to define two parity subspaces, $\{|\pm,j\rangle\}$, such that $\hat{\Pi}|\pm,j\rangle=\pm|\pm,j\rangle$ with $|+,j\rangle=(\hat{n}^{-1/2}\hat{a}^{\dagger}\hat{\sigma}_x)^j|0,e\rangle$ and $|-,j\rangle=(\hat{n}^{-1/2}\hat{a}^{\dagger}\hat{\sigma}_x)^j|0,g\rangle$; the states $|0,g\rangle$ and $|0,e\rangle$ correspond to the field in the vacuum state and the qubit in the ground or excited level, in that order. Thus, Eq. (4) becomes the Hamiltonians

$$\hat{H}'_{\pm}=\omega\hat{K}_0\pm\frac{\omega_0}{2}(-1)^{\hat{K}_0}+g(\hat{K}_+ +\hat{K}_-)$$
$$-\omega k\pm\frac{\omega_0}{2}(-1)^{-k},\qquad k>0 \tag{5}$$

in each parity subspace after defining $\hat{K}_0=\hat{n}+k$, $\hat{K}_+=\hat{a}^{\dagger}\sqrt{\hat{n}+2k}\hat{\sigma}_x$, and $\hat{K}_-=\sqrt{\hat{n}+2k}\hat{a}\hat{\sigma}_x$ such that they form the SU(1,1) group, $[\hat{K}_+,\hat{K}_-]=-2\hat{K}_0$ and $[\hat{K}_0,\hat{K}_\pm]=\pm\hat{K}_\pm$ [33,34]. In such a case, we can put aside the constant terms and focus on the parity subspace Hamiltonians

$$\hat{H}_{\pm}=\omega\hat{K}_0\pm\frac{\omega_0}{2}(-1)^{\hat{K}_0}+g(\hat{K}_+ +\hat{K}_-). \tag{6}$$

While in QED it may not make sense, in photonic lattices it is useful to define two regimes where the model is soluble using the qubit frequency as reference: (i) a weak coupling regime where the coupling constant is negligible compared to the qubit frequency, $g\ll\omega_0$, where in the case $g/\omega_0\to 0$ the eigenstates of the model are the parity states $|\pm,j\rangle$ with energy $E_{\pm,j}=\omega(j+k)\pm\omega_0(-1)^{j+k}/2$ and (ii) a deep-strong coupling regime where the qubit frequency is negligible compared to the coupling constant [35,36], $g\gg\omega_0$, where in the case $\omega_0/g\to 0$ the eigenstates are $su(1,1)$ generalized coherent states, $|\pm,\xi\rangle=S(\xi)|\pm,j\rangle$, with energy $(\omega^2-4g^2)^{1/2}(j+k)$. The unitary displacement is given by $S(\xi)=e^{-\xi(\hat{K}_+-\hat{K}_-)/2}$ [37,38], with $\tanh\xi=2g/\omega$ for our case; note that the displacement parameter, $\xi=\arctan(2g/\omega)$, restricts the coupling values for this regime to $g<\omega/2$. At this point, we can follow an argument identical to that found in [18] and find that despite the fact that the modified evolution operator $\hat{U}_{\pm}=e^{-i(\hat{H}_{\pm}+\omega_0/2)t}$ is apparently unitary, the value of $\langle\pm,j|\hat{U}_{\pm}|\pm,k\rangle$ diverges at any finite time for $g\geq\omega/2$ and, thus, the model seems to be valid just for values of $g<\omega/2$. While we are not able to discuss a physical reason behind this cutoff due to the artificial nature of the model, it may be related to a topological issue in the definition of generalized coherent states [19–21], e.g., in the case $g=\omega/2$, it is possible to find an invertible transformation, $e^{\hat{K}_+}$, that is not unitary but rotates $\omega(\hat{K}_0+\hat{K}_x)$ into $\hat{K}_-$, and, then, the eigenstates can be produced as generalized coherent states à la Barut–Girardello [39], $\hat{K}_-|j,\alpha\rangle=\alpha|j,\alpha\rangle$.

In any given set of frequencies and coupling parameters, e.g., $\{\omega,\omega_0,g\in[0,\omega/2]\}$, the model in the parity bases becomes a tridiagonal, real, symmetric, semi-infinite matrix whose eigenvalues and eigenvectors can be approximated by standard linear algebra methods or discussed analytically following standard methods [40,41]. Figure 1 shows numerically calculated spectra in the positive and negative parity subspaces for the model Hamiltonian $\hat{H}_\pm$ as a function of the qubit frequency. The equidistant behavior predicted for the extremes of the weak and deep-strong coupling regimes can already be observed. The spectra show avoided crossings between the energies of a given parity and crossings between energies of different parity in a similar manner as the spectra of the quantum Rabi Hamiltonian, where integrability has been argued on this basis [40].

3. SUPERSYMMETRY IN THE REDUCED SU(1,1) MODEL

Let us consider the limiting case of the deep-strong coupling regime where $\omega_0=0$; again, this may not make sense while thinking of cavity-, trapped-ion-, or circuit-*QED*, but such a model can be produced in photonic lattices [11] and define the unperturbed Hamiltonian

$$\hat{H}_0=\omega\hat{K}_0+g(\hat{K}_+ +\hat{K}_-). \tag{7}$$

If we define a qubit-field annihilation (creation) operator as $\hat{A}=\alpha\hat{a}+(g/\alpha)\sqrt{\hat{n}+2k}\hat{\sigma}_x$ $(\hat{A}^{\dagger}=\alpha\hat{a}^{\dagger}+(g/\alpha)\sqrt{\hat{n}+2k}\hat{\sigma}_x)$ with parameter $\alpha^2=(\omega+\sqrt{\omega^2-4g^2})/2$ where the restriction $g<\omega/2$ appears once more, we can write two unbroken SUSY partners:

$$\hat{A}^{\dagger}\hat{A}=\omega\hat{K}_0+g(\hat{K}_+ +\hat{K}_-)-k\sqrt{\omega^2-4g^2}, \tag{8}$$

$$\hat{A}\hat{A}^{\dagger}=\omega\tilde{K}_0+g(\tilde{K}_+ +\tilde{K}_-)+\left(\frac{1}{2}-k\right)\sqrt{\omega^2-4g^2}, \tag{9}$$

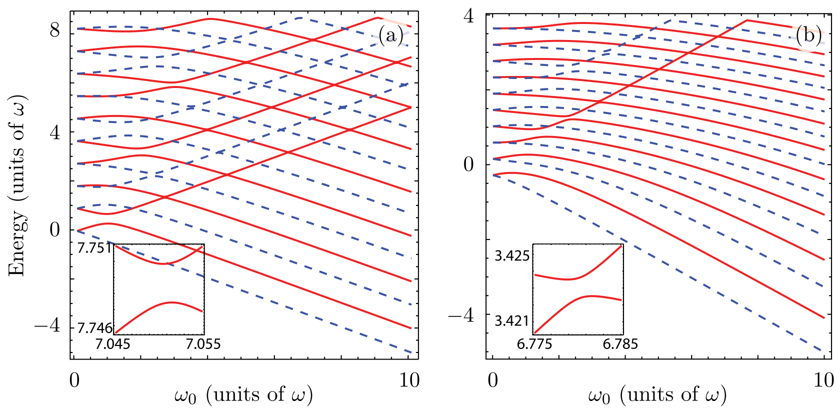

Fig. 1. Spectra for the positive (red solid lines) and negative (blue dashed lines) parity subspaces of the model $\hat{H}'_\pm$ with $k=1/2$, that is, the BSM plus counter-rotating terms, for variable qubit frequency ω_0 with a fixed coupling parameter (a) $g=0.2\omega$ and (b) $g=0.45\omega$. The insets show typical avoided crossings.

where the tilded operators are a different representation of SU(1,1): $\tilde{K}_0 = \hat{n} + k + 1/2$, $\tilde{K}_+ = \sqrt{\hat{n} + 2k}\hat{a}^\dagger \hat{\sigma}_x$, and $\tilde{K}_- = \hat{a}\sqrt{\hat{n} + 2k}\,\hat{\sigma}_x$. Note that both partners are covered by the initial Hamiltonian, Eq. (4), for a degenerate qubit because $\tilde{K}_+ = \hat{a}^\dagger\sqrt{\hat{n} + 2k + 1}\hat{\sigma}_x$ and $\tilde{K}_- = \sqrt{\hat{n} + 2k + 1}\hat{a}\hat{\sigma}_x$. The two SUSY partners are diagonalized by the displacement $S(\xi)$ defined before and reduce to the following form:

$$S(-\xi)\hat{A}^\dagger\hat{A}S(\xi) = \sqrt{\omega^2 - 4g^2}\hat{n}, \qquad \Omega_j = \sqrt{\omega^2 - 4g^2}j, \tag{10}$$

$$S(-\xi)\hat{A}\hat{A}^\dagger S(\xi) = \sqrt{\omega^2 - 4g^2}(\hat{n} + 1), \qquad \Omega_j^{(p)} = \sqrt{\omega^2 - 4g^2}(j + 1), \tag{11}$$

where it is possible to realize that their spectra are identical, $\Omega_k = \Omega_{k-1}^{(p)}$. A particular case of such unbroken SUSY partners has been previously discussed for the parameter set $k = 1/2$, $k = 1$, $\omega = 1 - \alpha^2$, and $g = -\alpha$, with $\alpha \neq 1$ in the context of photonic isospectral lattices [32].

4. OPTICAL SIMULATION

The optical simulation of the quantum Rabi model in arrays of coupled waveguides inscribed by laser damage in fused silica has been proposed and demonstrated experimentally [42]. Nonlinear quantum Rabi models are also feasible for optical simulation [11] if care is exerted on the validity of the Hamiltonians and the characteristics of the required lattices [43]. To produce the lattice, we follow a standard procedure, which in our case means inserting the general state $|\Psi_\pm\rangle = \sum_{j=0}^{\infty} \mathcal{E}_j^{(\pm)}|\pm, j\rangle$ and the Hamiltonian $\hat{H}'_\pm$ into the Schrödinger equation and making the change of variable $t \to -z$ to obtain the differential equation sets

$$-i\partial_z \mathcal{E}_j^{(\pm)} = n_j^{(\pm)}\mathcal{E}_j^{(\pm)} + \gamma_{j-1}\mathcal{E}_{j-1}^{(\pm)} + \gamma_j\mathcal{E}_{j+1}^{(\pm)}, \qquad \mathcal{E}_{-|j|} = 0. \tag{12}$$

These sets describe a tight-binding photonic lattice where the effective refractive index of the jth waveguide is given by $n_j^{(\pm)} = \omega j \pm \omega_0(-1)^j/2$, up to a constant bias refractive index shared by all waveguides, and the coupling between neighbor jth and $(j+1)$th waveguides is given by $\gamma_j = g\sqrt{(j+1)(j+2k)}$, with the Bargmann parameter $k > 0$ and the restriction $g < \omega/2$, as discussed before. The generalities of the optical simulation of quantum phenomena can be found in reviews on the topic [22,23,44,45]. We want to stress that while the theoretical quantum-optical model requires a semi-infinite array of coupled waveguides, it is possible to cut off the size of the array depending on the initial state to propagate. This cutoff also helps in keeping the photonic lattice experimentally feasible, as stronger coupling parameter values require closer waveguides that may prove a complication in the laboratory and produce coupling between second- or higher-order neighbors.

Note that in an optical simulation realized with a classical field propagating through an array of coupled waveguides, it is trivial to measure the light intensity in each waveguide at the end of the photonic array. Thus, the mean photon number $\langle\hat{n}(z)\rangle = \sum_j j|\mathcal{E}_j^{(+)}(z)|^2$, which is equivalent to the barycenter of the intensity, and the mean atomic excitation energy $\langle\hat{\sigma}_z(z)\rangle = \sum_j[|\mathcal{E}_{2j+1}^{(+)}(z)|^2 - |\mathcal{E}_{2j}^{(+)}(z)|^2]$, which is equivalent to the difference between the total intensity at odd and even waveguides, are feasible quantum optical measurements to be recovered by the optical simulation. Figure 2 shows the propagation of the initial state $|\psi(0)\rangle = |0, e\rangle$ under the dynamics imposed by $\hat{H}'_+$ as a classical simulation provided by light impinging the zeroth waveguide of a photonic lattice with the parameter set $\{\omega, \omega_0, g, k\} = \{\omega, 3\omega/4, 2\omega/5, 1/2\}$ and a lattice size of 200 waveguides. Quasi-periodical $\sim 10\pi$ returns to a state close to the initial state can be observed in the intensity of the zeroth waveguide, $|\mathcal{E}_0(z)|^2$, mean photon number, $\langle\hat{n}(z)\rangle$, and mean atomic excitation energy, $\langle\hat{\sigma}_z(z)\rangle$. This is an interesting phenomenon that we were not expecting in the model for such a high coupling parameter and should be probed in the future.

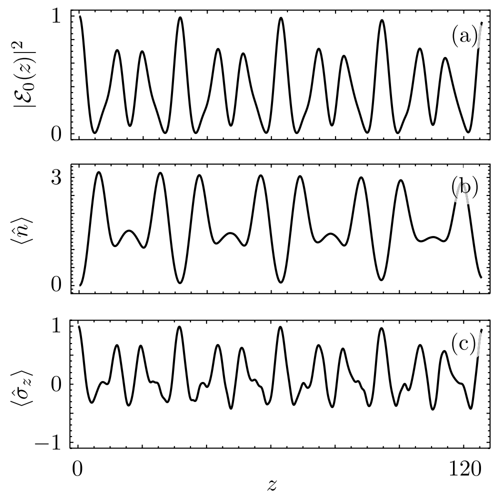

Fig. 2. Numerical simulation of evolution under $\hat{H}'_+$ with an initial state $|0, e\rangle$ and parameter set $\{\omega, \omega_0, g, k\} = \{\omega, 3\omega/4, 4\omega/10, 1/2\}$. (a) Intensity at the zeroth waveguide, (b) mean photon number equivalent to the intensity barycenter, and (c) mean atomic excitation energy equivalent to the total intensity in odd waveguides minus the total intensity in even waveguides.

It is important to note that the proposed optical simulation will not be ideal. Actually, the almost negligible losses from each of the waveguides to the environment allow us to follow the propagation of classical fields through the photonic array by imaging the plane of propagation [27]. While these losses may be engineered in the photonic crystal, they are not easily related to the effect of dissipation in the quantum optics model [46,47], but it may be possible to take a dissipative model and engineer an adequate classical or quantum simulator [44,48].

5. CONCLUSION

We have proposed an intensity-dependent quantum Rabi model with an underlying parity and SU(1,1) symmetry. In the case $k = 1/2$, our model reduces to the BSM plus counter-rotating terms. As expected from the behavior of the BSM, our model seems to be invalid for coupling factors of $g \geq \omega/2$. The behavior of the spectra is similar to the quantum Rabi model, that is, avoided crossings in spectral branches belonging to the same parity and crossings between spectral branches belonging to different parities. In the special case of degenerate qubit frequency, $\omega_0 = 0$, it is straightforward to diagonalize the model with generalized SU(1,1) coherent states. It is also possible to provide qubit-field creation and annihilation operators that fulfill the commutator for the field and allow us to construct an unbroken SUSY partner for it; the SUSY partners correspond to Bargmann parameters k and $k_p = k + 1/2$. This gives a recipe to a class of isospectral photonic lattices. Finally, we discussed the classical simulation of the full theoretical model in finite arrays of coupled photonic waveguides and showed by numerical simulation that it is possible to find quasi-periodic reconstruction for a given initial state in the full intensity-dependent quantum Rabi model for a given parameter set.

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