

Full characterization of Airy beams under physical principles

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The propagation characteristics of Airy beams is investigated and fully described under the traveling-wave approach analogous to that used for nondiffracting Bessel beams. This is possible when noticing that Airy functions are, in fact, Bessel functions of fractional order $\frac{1}{3}$. We show how physical principles impose restrictions such that the nondiffracting Airy beams cannot be of infinite extent as has been argued and introduce quantitative expressions for the maximum transverse and longitudinal extent of Airy beams. We show that under the appropriate physical conditions it is possible to obtain higher-order Airy beams.

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I. INTRODUCTION

Bessel beams are known to belong to a class of optical beams described by the $(2 + 1)$ -dimensional Helmholtz equation, and they have the property of being nondiffracting [1,2] and self-healing [3,4] on propagation. Originally, Bessel beam properties were studied using the diffraction Rayleigh-Sommerfeld or Fresnel-Kirchhoff integral. Implicitly, this approach considers the Bessel beam to be of infinite transverse extent [1]. Alternatively, a more straightforward and physically sound description of these beams can be done in terms of traveling Hankel waves [2]. This description based on the differential Helmholtz wave equation provides a better understanding of the physical origin of these beams and their intriguing properties. Even further, this same approach allowed us to demonstrate the existence of other families of nondiffracting beams described by fundamental solutions of the separable $(2 + 1)$ Helmholtz wave equation. These families of beams show propagation characteristics analogous to the Bessel beams [5,6].

In recent years, another kind of nondiffracting beam has been reported, but its propagation is governed by the paraxial wave equation in $(1 + 1)$ dimensions, the so-called Airy beam [7]. These beams do not belong to the same class as those of the Helmholtz equation, and thus it might be expected that their physical properties are not actually described in the same terms. Nonetheless, an interesting fact that is hardly discussed in all the published literature on Airy beams is that the Airy functions are Bessel functions of fractional order equal to $1/3$. This allows the application of the aforementioned traveling Hankel wave description. In this work we show that Airy beams have properties similar to those of Bessel

beams because the former are the result of the superposition of counterpropagating Hankel traveling waves of fractional order. We also demonstrate that the proposed wave approach imposes the condition under physical principles for these beams to be of finite extent, contrary to the “ideal” infinite Airy beam.

The aim of this paper is to provide a better understanding of the fundamental nature of the traveling-wave methodology applied to Airy beams. In particular we show that all the known propagation characteristics of Airy beams are straightforwardly and intuitively understood using the Hankel traveling-wave approach and that the focusing features of $(1 + 1)$ -dimensional Airy beams can only be described in clear and simple terms with this methodology, which is similar to that for $(2 + 1)$ -dimensional Bessel beams. In Sec. II we provide a brief account of the traveling-wave description for Bessel beams as a way to motivate its application to Airy beams. Section III explains the relationship between Airy and Bessel functions, and in Sec. IV we show the characteristics of the finite Airy beam in a direct manner. Finally, in Sec. V we show that higher-order Airy beams can exist by discussing the physical conditions that enable this possibility.

II. TRAVELING HANKEL WAVES AND BESSEL BEAMS

Light propagation in linear media is described with the use of the scalar wave equation for the electric field $E(\mathbf{r}, t)$ given by

$$\nabla^2 E(\mathbf{r}, t) = \frac{1}{v^2} \frac{\partial^2 E(\mathbf{r}, t)}{\partial t^2}. \quad (1)$$

In cylindrical coordinates $\mathbf{r} = (r, z)$, and v corresponds to the speed of light in the medium in question. It is possible to solve Eq. (1) by separation of variables [8], resulting in ordinary differential equations for the variables r , z , and t . The separation constants can be chosen such that $\omega^2/v^2 = k_r^2 + k_z^2 \equiv |\mathbf{k}|^2$, so that we can think of \mathbf{k} as the wave vector. In

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this manner, the equation for the radial coordinate from Eq. (1) is given by

$$\frac{d^2 H}{dr^2} + \frac{1}{r} \frac{dH}{dr} + \left(k_r^2 - \frac{m^2}{r^2} \right) H = 0, \quad (2)$$

which is the Bessel differential equation of order m [9], and its solutions are given by the m th order Bessel function $J_m(k_r r)$ and the m th order Neumann function $N_m(k_r r)$. In many cases the Neumann functions mentioned above are discarded due to the singularity they present at the origin. However, it has been proved by one of the authors [2] that these functions do indeed carry physical meaning. Also, the solutions J_m and N_m cannot be used separately to describe the *propagation* of light because they do not satisfy the Sommerfeld radiation condition independently. For cylindrical waves this condition reads

$$\lim_{r \rightarrow \infty} r^{1/2} \left(\frac{dH}{dr} - ik_r H \right) = 0, \quad (3)$$

and it tells us that a wave equation cannot have waves coming from an infinite distance. Thus, the solution needed to describe *propagating* waves must be given by the complex superposition of the J_m and N_m functions. This leads to the so-called Hankel waves [2],

$$H_m^{(1)}(k_r r) = J_m(k_r r) + iN_m(k_r r), \quad (4)$$

$$H_m^{(2)}(k_r r) = J_m(k_r r) - iN_m(k_r r). \quad (5)$$

Once the azimuthal and longitudinal wave components are incorporated, $\exp(im\varphi + ik_z z)$, the wave fronts of the solutions for the Helmholtz wave equation in cylindrical coordinates are conic helicoids. $H_m^{(1)}$ and $H_m^{(2)}$ are related to outgoing (convex) and incoming (concave) conic solutions, respectively. The Hankel waves, in combination with the temporal part, describe cylindrical wave fronts that collapse and are generated at the longitudinal z axis. This is the physical origin of the singularity of the Hankel functions; the longitudinal axis is simultaneously the sink and source for the incoming and outgoing cylindrical wave components, respectively. In that sense, since incoming waves become outgoing, there is a region where both of them interfere, leaving only the Bessel function:

$$\begin{aligned} E_{\text{in}}(r, \varphi, z, t) + E_{\text{out}}(r, \varphi, z, t) \\ = 2J_m(k_r r) \exp(im\varphi + ik_z z - i\omega t). \end{aligned} \quad (6)$$

Recalling the Sommerfeld radiation condition, Eq. (3), the incoming wave must be generated at a finite distance, implying that Bessel beams must have finite transverse extent and thus a finite propagation distance. The interference of both Hankel waves only occurs within a conic region, and it is only within this region that the Bessel beam can be formed. This region is therefore called *the region of existence* of the Bessel beam, which can be exploited in applications such as the design of laser resonators whose output is related to a Bessel beam [10].

As we have mentioned before, Bessel beams are nondiffractive; that is, they propagate without spreading or changing their shape within their region of existence. Similarly, they show the property of self-healing that occurs when the beam is partially

blocked; that is, it reconstructs itself after some distance [4]. These properties can be straightforwardly understood in terms of traveling Hankel waves, which provides a clear frame for the physics behind them and others like the evolution of focused Bessel beams [11].

III. TRAVELING-WAVE APPROACH TO AIRY BEAMS

Berry and Balazs introduced the idea of the “nonspreading Airy wave packet” by solving a force-free Schrödinger equation [12]. The solution propagates without change along a parabolic trajectory and thus shows acceleration. The relationship between the force-free Schrödinger equation and the paraxial wave equation indicates that nondiffractive Airy beams are therefore possible, and indeed, they have been observed [7]. Furthermore, it has been reported that, similar to Bessel beams, Airy beams also have the property of self-healing [13].

Airy beams can be obtained when considering the propagation of a plane-polarized beam in a linear medium described by the normalized paraxial wave equation,

$$-i \frac{\partial U}{\partial \xi} + \frac{1}{2} \frac{\partial^2 U}{\partial s^2} = 0, \quad (7)$$

where $U(s, \xi)$ is the electric-field envelope that depends on the normalized coordinates $s = x/x_0$ and $\xi = z/kx_0^2$, with x_0 being a given transverse scale and k being the wave number. Since we know that the Airy beam remains invariant while propagating along a curved trajectory, we can define an accelerating variable $\Gamma = s - \frac{a}{4}\xi^2 + v\xi$, with a and v being real constants. We can now write the electric field envelope as $U(s, \xi) = w(\Gamma) \exp[i\theta(\Gamma, \xi)]$, which leads to the following ordinary differential equation for $w(\Gamma)$:

$$w'' - \alpha \Gamma w = \frac{A^2}{w^3}, \quad (8)$$

and $\theta(\Gamma, \xi)$ is given by

$$\theta(\Gamma, \xi) = A \int_0^\Gamma \frac{d\Gamma'}{w^2} + \left(\frac{a}{2}\xi - v \right) + \frac{a^2}{24}\xi^3 - \frac{av}{4}\xi^2 + \frac{v}{2}\xi, \quad (9)$$

where $A = A(\xi)$ is assumed to be a constant. We can think of parameters a and v as the acceleration and velocity of the beam, respectively. Without loss of generality, let us consider the case where $a = 1$, $v = 0$, and $A = 0$; Eq. (8) becomes

$$w'' - \left(s - \frac{1}{4}\xi^2 \right) w = 0, \quad (10)$$

which is the well-known Airy differential equation [14], and therefore, the solution can be expressed as

$$U(s, \xi) = \text{Ai} \left(s - \frac{\xi^2}{4} \right) \exp \left[i \left(\frac{s\xi}{2} - \frac{\xi^3}{12} \right) \right], \quad (11)$$

where $\text{Ai}(\cdot)$ is the Airy function. The expression above implies that the intensity of the beam has the profile of the modulus squared of the Airy function, i.e.,

$$I(s, \xi) = \left| \text{Ai} \left(s - \frac{\xi^2}{4} \right) \right|^2. \quad (12)$$

The argument of the Airy function in Eq. (11) shows that the Airy beam follows a parabolic trajectory that can be interpreted as having a transverse acceleration [12]. It is also clear that it does not change either its profile or its amplitude on propagation when observed along this parabolic trajectory. In this sense the Airy beam can be regarded as being a nondiffracting beam, as is the case for the Bessel beam discussed in Sec. II. It may seem that although both Airy and Bessel beams are nondiffracting, the physics are not described in a similar way given that they do not belong to the same class of beams. However, in what follows we will show that Airy beams can be better understood when described by the traveling-wave approach used in the treatment of Bessel beams outlined in Sec. II.

In order to elucidate the properties and behavior of Airy beams, let us recall Airy's differential equation, namely,

$$\frac{d^2 w}{ds^2} \mp s w = 0, \quad (13)$$

whose solutions are given by the Airy functions $\text{Ai}(s)$ and $\text{Bi}(s)$ [14], where we are using normalized coordinates. The $\text{Bi}(s)$ function is defined as the solution with the same amplitude of oscillation as $\text{Ai}(s)$ as s goes to minus infinity and differs in phase by $\pi/2$. We will therefore concentrate on the behavior of $\text{Ai}(s)$ in the rest of this discussion. The Airy differential equation is defined in the entire real space, and thus the second term in Eq. (13) can have either a positive or a negative sign depending on whether the value of s is positive or negative. Using either of the two signs yields the same Airy solution with the only difference being that one will be the mirror reflection of the other with respect to the vertical axis, $s = 0$.

A simple calculation shows that by making the change of variable $w = \sqrt{s} Z_{1/3}(\frac{2}{3}s^{3/2})$, Eq. (13) can be transformed into the Bessel differential equation of order $1/3$, with $Z_{1/3}$ being the cylindrical Bessel function of order $1/3$ [15,16]. In a similar way, the Airy functions can be expressed in terms of modified Bessel functions of order $1/3$, $K_{1/3}$, as follows [14]:

$$w(s) = \frac{1}{\pi} \sqrt{\frac{s}{3}} K_{1/3} \left(\frac{2}{3} s^{3/2} \right), \quad (14)$$

$$w(s) = \frac{1}{2} \sqrt{\frac{s}{3}} e^{2\pi i/3} H_{1/3}^{(1)} \left(\frac{2}{3} i s^{3/2} \right). \quad (15)$$

We can distinguish two important cases; let us consider the argument of these Bessel functions $K_{1/3}$ to be $\chi = \frac{2}{3}|s|^{3/2}$. On the one hand, when $s \geq 0$, the profile has a monotonic decreasing behavior and is proportional to the modified Bessel function $K_{1/3}$:

$$\text{Ai}(s) = \frac{1}{\pi} \sqrt{\frac{s}{3}} K_{1/3}(\chi). \quad (16)$$

On the other hand, when $s < 0$, the profile can be considered the superposition of two waves which are essentially Hankel functions of order $1/3$, i.e.,

$$\text{Ai}(-s) = \frac{1}{2} \sqrt{\frac{s}{3}} \left[e^{i\frac{\pi}{6}} H_{1/3}^{(1)}(\chi) + e^{-i\frac{\pi}{6}} H_{1/3}^{(2)}(\chi) \right]. \quad (17)$$

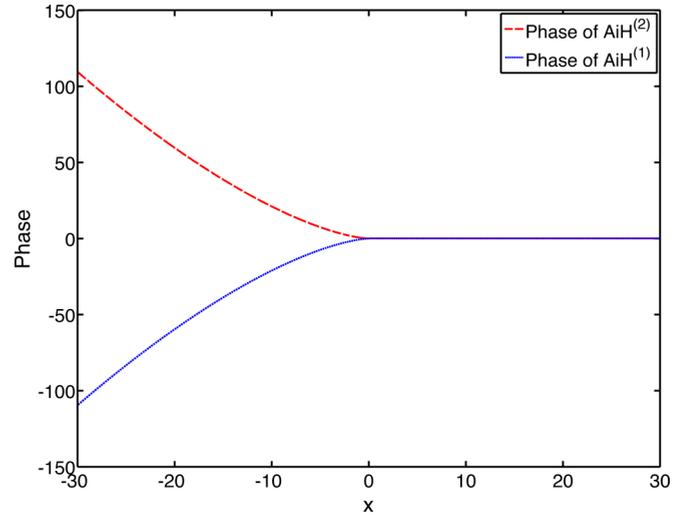


FIG. 1. (Color online) Schematic wave fronts of the $\text{AiH}^{(1)}(s)$ and $\text{AiH}^{(2)}(s)$ functions (arbitrary units).

We can now define the following functions:

$$\text{AiH}^{(1)}(s) = \frac{1}{2} \sqrt{\frac{s}{3}} e^{i\frac{\pi}{6}} H_{1/3}^{(1)}(\chi), \quad (18)$$

$$\text{AiH}^{(2)}(s) = \frac{1}{2} \sqrt{\frac{s}{3}} e^{-i\frac{\pi}{6}} H_{1/3}^{(2)}(\chi). \quad (19)$$

With these definitions, we can now write the Airy function in a more compact form as follows:

$$\text{Ai}(-s) = \text{AiH}^{(1)}(s) + \text{AiH}^{(2)}(s). \quad (20)$$

We identify these functions as the traveling Hankel components of the Airy beam, in analogy to the way it was done for Bessel beams in Sec. II. We note that it is possible to obtain an analytical form of the phase of these Hankel components by considering the asymptotic expansion of Hankel functions, which approximate very well to the original function from the first maximum [14]. Using this expansion, we have that

$$\text{AiH}^{(1)}(s) \simeq \sqrt{\frac{s}{6\pi\chi}} e^{i\phi}, \quad (21)$$

$$\text{AiH}^{(2)}(s) \simeq \sqrt{\frac{s}{6\pi\chi}} e^{-i\phi}, \quad (22)$$

with $\phi = \chi - \frac{\pi}{4}$ and their wave fronts being opposite each other. In Fig. 1, we can see the wave fronts of the components $\text{AiH}^{(1)}(s)$ and $\text{AiH}^{(2)}(s)$. Each phase determines a geometric wave front, and thus according to geometrical optics, the rays that make up each beam will propagate perpendicularly to this front [17]. In fact, when the Airy beam is cut, it will propagate along trajectories determined by the rays of the $\text{AiH}^{(2)}(s)$ beam. We note that these waves must satisfy the Sommerfeld radiation condition that imposes the restriction of Airy beams being of finite extent, and similar to Bessel beams, they can only exist within a finite region of space.

In order to show that it is, in fact, the $\text{AiH}^{(2)}$ component that bears the property of the parabolic caustics we carried out the propagation of each component independently. In Fig. 2(a) we see the propagation of the $\text{AiH}^{(1)}$ component that

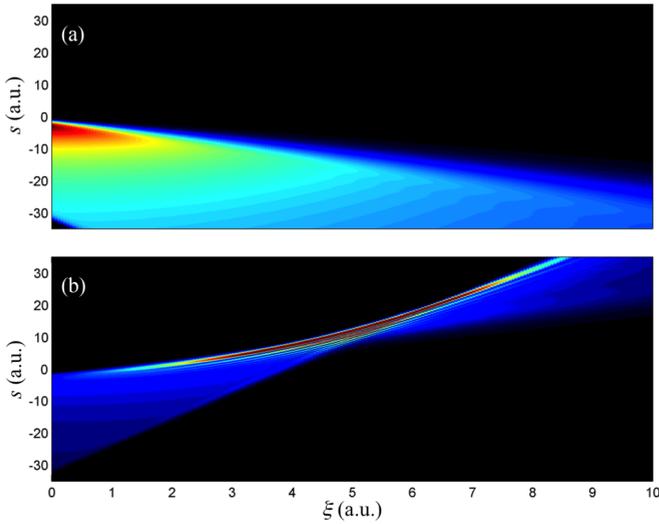


FIG. 2. (Color online) Behavior of the independent propagation of (a) the $\text{Ai}H^{(1)}$ and (b) the $\text{Ai}H^{(2)}$ components of the Airy beam in arbitrary dimensionless units.

simply travels away from the propagation axis, diminishing its amplitude. In Fig. 2(b) the parabolic caustic associated with the rays normal to the concave wavefront can be appreciated. We note that the caustic determined by the rays defines the parabolic motion of the main maximum of the beam under consideration [12]. This behavior is similar to that produced by a third-order aberration (coma). The numerical simulations shown in this paper use a computational window such that the dimensionless parameters are $s \in [-30, 30]$ and $\xi \in [0, 5]$, and we have used a super-Gaussian window $t(x) = \exp -(s/t_0)^{50}$, where t_0 is the width of the window, to reduce the diffraction effects introduced if a hard aperture is used instead and to emphasize the propagation features of the AiH beams.

We can now easily understand the self-healing property of the Airy beam: since the Airy beam is formed by the

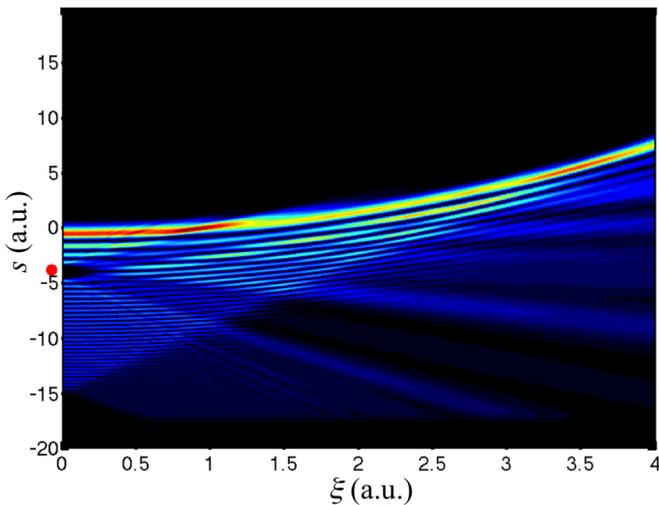


FIG. 3. (Color online) Propagation of an obstructed Airy beam (in arbitrary dimensionless units) showing self-healing. Notice the presence of two shadows. The red dot at the edge of the figure indicates the position where the obstruction has been located.

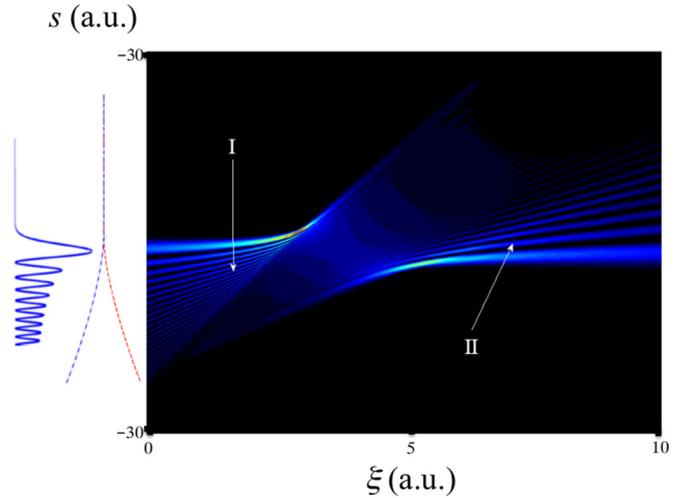


FIG. 4. (Color online) Behavior of a focused Airy beam (in arbitrary dimensionless units); observe the two regions due to the focusing of the two Hankel traveling waves. On the left side we have indicated the profile of the Airy beam as well as the two wave fronts of the beam.

superposition of the fractional Hankel components $\text{Ai}H^{(1)}(s)$ and $\text{Ai}H^{(2)}(s)$, the self-healing region arises from the recombination of these two components after the shadows of the obstructed waves, as seen in Fig. 3.

In regard to focusing, we note that the Airy beam shows the noncommon behavior of presenting two different focusing regions, one for $\text{Ai}H^{(2)}(s)$ in which focusing is observed and a second for $\text{Ai}H^{(1)}(s)$, farther away, where a defocusing beam appears. This behavior can be seen in Fig. 4, where we have marked the two regions as I and II. This behavior is easily explained by noting that the composing Hankel waves, besides having opposite traveling directions, have opposite wave-front curvatures: one is positive focusing and the other is negative defocusing. When passing through the positive lens, these add or subtract accordingly.

A simple geometrical analysis under this consideration easily explains the two observed regions. In Fig. 5 we show a schematic of the optical arrangements including a lens; the wave fronts have been marked in two different colors. Notice how each region is generated by the focusing of the rays coming from each of the two wave fronts, i.e., from each of the Hankel waves that compose the Airy beam. We can think of each region being formed due to the simultaneous incidence of each Hankel component of the Airy beam. We would like to

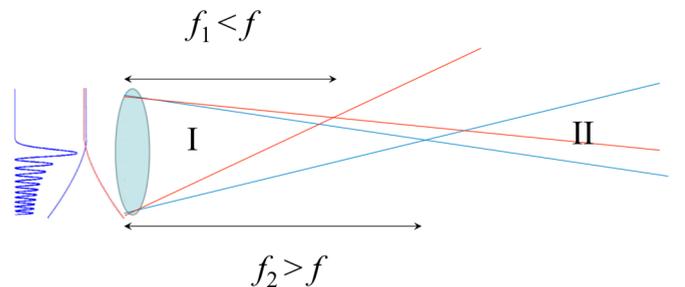


FIG. 5. (Color online) Schematic ray tracing of the double focusing of an Airy beam related to that shown in Fig. 4.

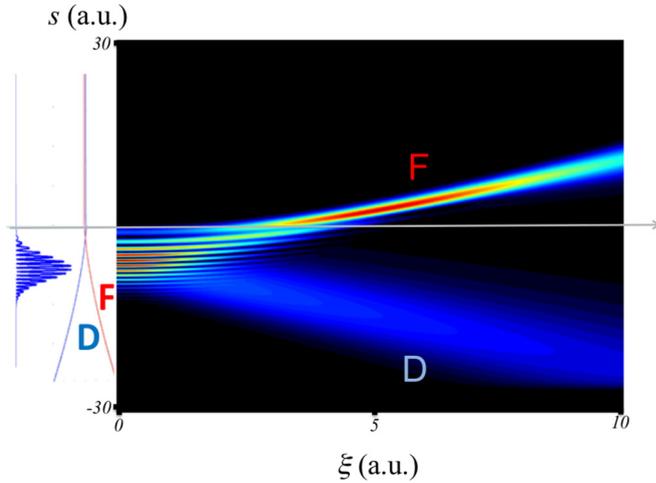


FIG. 6. (Color online) Propagation of an off-axis Airy-Gauss beam using arbitrary dimensionless units

emphasize that this is the case regardless of any interference pattern caused by the functions $\text{Ai}H^{(1)}$ and $\text{Ai}H^{(2)}$. However, we can indeed go further and assume that it is possible to make use of a packet that is either only the real or only the imaginary part of one of these two components, for example, $\text{Ai}H^{(1)}$.

Finally, we can now take a more general view of the behavior of an Airy-Gauss beam when it is off axis. In Fig. 6 we show the propagation of the Airy-Gauss beam, where it is clear that each of the Hankel components provides the two contrasting behaviors of focusing and defocusing mentioned before.

As we have seen, the application of the traveling Hankel wave approach has provided us with a straightforward description of the propagation characteristics of the Airy beam under different circumstances. We will now show how to determine the propagation distance of finite-energy Airy beams.

IV. FINITE AIRY BEAMS

It is usually argued that nondiffracting beams, such as the Airy beam, require an infinite amount of energy for their generation. This might be to give an explanation for the nondiffraction feature or because the range of the mathematical function that describes the profile is infinite. However, in this section we show that from a physical perspective this cannot be the case as the beam must be constrained by physical principles. One of the principles mentioned above for the Bessel beams also applies for the Airy beams, and that is the Sommerfeld radiation condition. Within the traveling wave approach, to have an infinite Airy beam would require having sources at infinity.

For Airy beams, whose profile exhibits reduction of the separation distance between intensity peaks, there exists another more basic physical constraint that we discuss now. The distance between two consecutive peaks in the transverse Airy intensity profile should not become smaller than the wavelength of the light used. If that were the case, we would end up with an unphysical situation. This situation is analogous to the treatment, for instance, of wave excitations with a fractal boundary [18,19]; although mathematically the fractal

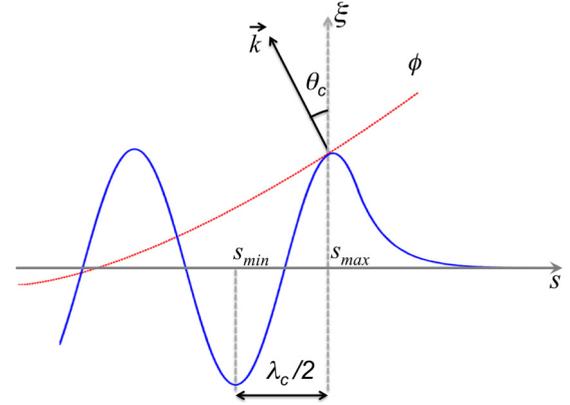


FIG. 7. (Color online) Schematic of the behavior of an Airy beam: When the wave vector \vec{k} of the Airy beam, which is normal to the wave front ϕ , reaches the maximum paraxial value $\theta_c = \frac{\pi}{6}$, the oscillations in the profile must dampen in order to remain physical. At that point, the distance between two extreme points in the profile must be of the order of half the critical wavelength λ_c .

structure continues to infinitely small scales, in reality there are physical constraints that avoid this situation.

Next, we proceed to find a critical value of the beam extent after which its profile must dampen, giving rise to a finite-energy Airy beam. In order to tackle the issue, let us take the asymptotic expansion for large arguments of the Airy function given by the corresponding Bessel functions of order $1/3$:

$$\text{Ai}(\alpha s) \propto \cos\left(\frac{2}{3}(\alpha s)^{\frac{3}{2}} - \frac{\pi}{4}\right), \quad (23)$$

where we have taken the normalized coordinates and α is a parameter that allows us to change the frequency in the Airy function. We need to find the distance between two consecutive extreme points of the equation above (see Fig. 7). These points occur when the following conditions are met: For minima we have

$$s_{\min} = \frac{1}{\alpha} \left(\frac{3\pi}{2}\right)^{\frac{2}{3}} \left(2l - \frac{3}{4}\right)^{\frac{2}{3}}, \quad (24)$$

and for maxima we have

$$s_{\max} = \frac{1}{\alpha} \left(\frac{3\pi}{2}\right)^{\frac{2}{3}} \left(2l + \frac{1}{4}\right)^{\frac{2}{3}}, \quad (25)$$

where l is a nonnegative integer that provides us with information about the number of cycles that have occurred for a particular value of l .

In this way, the distance between two consecutive extremes is thus given by

$$\Delta s = \frac{1}{\alpha} \left(\frac{3\pi}{2}\right)^{\frac{2}{3}} \left[\left(2l - \frac{3}{4}\right)^{\frac{2}{3}} - \left(2l + \frac{1}{4}\right)^{\frac{2}{3}} \right]. \quad (26)$$

We know that the distance between two consecutive peaks in the beam intensity cannot be smaller than the transverse wavelength λ_c , i.e., $\Delta s \geq \lambda_c/2$. This argument provides a physical criterion for the cutoff point of a finite Airy beam after which it must dampen.

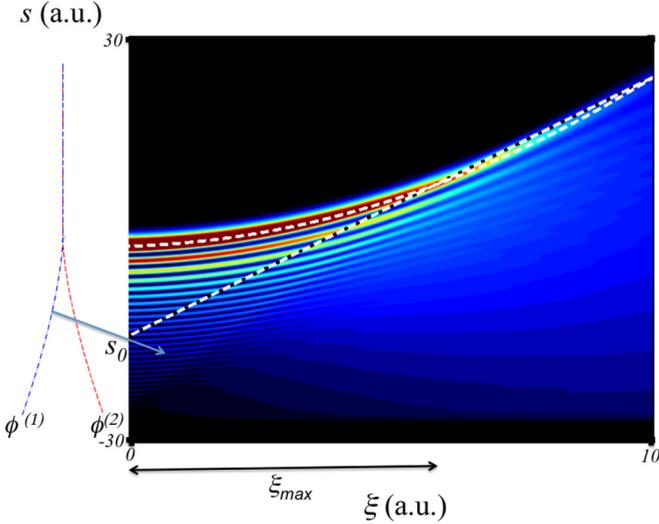


FIG. 8. (Color online) The method to determine the propagation distance ξ_{\max} of an Airy beam is given by the intersection of the parabolic trajectory followed by the main maximum and a ray coming from the edge of the profile (in arbitrary dimensionless units). We also show a diagram of the phases of the Hankel components of the Airy beam.

To define the critical transverse wavelength λ_c we must also take into consideration that Airy beams propagate within the paraxial regime as governed by Eq. (7). For this purpose we must provide the maximum angle allowed for a ray in the wave front to be considered paraxial. Curiously enough, to date in the literature there is not an established quantitative criterion for such an angle, and, as noted by Beck several decades ago, “The term paraxial rays is a relative one and to some extent a matter of arbitrary choice.” It will depend on the tolerance error that one plans to accept [20].

We will introduce the quantitative criterion of paraxiality put forward by Agrawal, Siegman, and others [21–23] where paraxial optical beams can be focused or can diverge at angles up to a maximum critical value of $\theta_c = \pi/6$. With this in mind, we consider a ray perpendicular to the wave front of $\text{Ai}H^{(2)}$ at the cutoff position. When the wave vector \vec{k} to this ray reaches the maximum paraxial angle $\theta_c = \pi/6$, recalling that $|\vec{k}| = 2\pi/\lambda$, a simple calculation gives $\lambda_c = 2\lambda$. Thus, substituting $\Delta s = 2\lambda$ in Eq. (26) and solving for l_c , we can get the position $s_{\max}(l_c)$, where the oscillations of the Airy profile must dampen.

A. Maximum propagation distance of finite Airy beams

We now propose a method for determining the propagation distance of the finite Airy beam based on the observations made above. Consider the parabolic trajectory followed by the main maximum of the Airy beam and take a ray coming from the wave front at the opposite extreme of the window and find the point of intersection. Figure 8 shows this situation; in order to guide the eye we have marked the region of existence of the Airy beam given by the ray as well as the parabolic trajectory followed by the beam (white dotted lines).

We know that for an Airy beam propagating in a homogeneous medium, the normalized solution is given by Eq. (11), and thus, it is clear that the trajectory of the main maximum satisfies the following equation:

$$s - \left(\frac{\xi}{2}\right)^2 = a_1, \quad (27)$$

where $a_1 = -1.0187297\dots$ is the first zero of the derivative of the Airy function, that is to say, the position of the first intensity maximum at the onset of propagation. Furthermore, we know that light rays are normal to the wave front, and therefore, in this case, the equation for the ray we are interested in can be expressed as

$$\xi = s_0^{-1/2}(s - s_0), \quad (28)$$

where s_0 is the point where the beam is truncated, in other words, where the edge of the window is located. Using Eqs. (27) and (28) for the ray, we have

$$\xi_{\pm} = 2(\sqrt{|s_0|} \pm \sqrt{2|s_0| - |a_1|}), \quad (29)$$

$$s_{\pm} = 3s_0 \pm 2\sqrt{|s_0|}\sqrt{2|s_0| - |a_1|}. \quad (30)$$

The solution we seek is the one with the negative sign because it is the first intersection between the parabola and the ray, i.e., the point where the propagation ends. In this way, the propagation distance is given by

$$\xi_{\max} = 2(\sqrt{|s_0|} - \sqrt{2|s_0| - |a_1|}). \quad (31)$$

This expression defines the maximum propagation distance for a finite Airy beam.

V. HIGHER-ORDER AIRY MODES IN GRADIENT-INDEX MEDIA

Up to now we have concentrated on the treatment of the Airy beam propagating in homogeneous media as a result of solving the normalized paraxial wave equation given by expression (7), and we have ended up with a solution given by the Airy function. However, the Airy functions also appear in the description of the propagating modes in inhomogeneous media, in particular in one-dimensional linear gradient-index media [24–26].

It is known that in sol-gel materials the gradient-index profile can be controlled by diffusion and precipitation of dopants, including quadratic, quartic, and higher-gradient profiles [27–29]. These materials are used to construct Wood lenses that are thin radial gradient index lenses with plane-parallel surfaces whose index of refraction is of the form $N = N_{00} + N_{10}r^2 + N_{20}r^4 + \dots$. With this antecedent, it could be possible to create a material whose index of refraction is the one-dimensional equivalent of a Wood lens.

In this section we investigate the propagating modes for power-law gradient-index media and show that they are what we have come to call higher-order Airy beams.

Let us start with the Helmholtz equation in (1+1) dimensions,

$$\nabla^2 E + k^2 E = 0, \quad (32)$$

and let the medium have a dispersion relation such that the wave number can be expressed as $k^2(x) = k_0^2 + \beta[-k_0 - x^n + \frac{\beta}{4}]$, where β is a parameter that depends on the medium. We can now propose an ansatz such that the electric field is given by $E(x, z) = U(x, z) \exp[i(k_0 - \frac{\beta}{2})z]$. In the paraxial approximation we have that

$$2i \left(k_0 - \frac{\beta}{2} \right) \frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial x^2} - \beta x^n U = 0. \quad (33)$$

If we require that $\beta = 2k_0$ the wave number is given by $k^2(x) = -2k_0 x^n$, and Eq. (33) is simplified into

$$\frac{\partial^2 U}{\partial x^2} - \beta x^n U = 0, \quad (34)$$

which corresponds to a generalized form of the Airy differential equation. Making the transformation [16]

$$U(x) = \sqrt{x} Z_{\frac{1}{n+2}} \left(\frac{2\sqrt{\beta}}{n+2} x^{\frac{n+2}{2}} \right) \quad (35)$$

and substituting it into Eq. (34), after some algebra, yield

$$Z''_{\frac{1}{n+2}}(\zeta) + \frac{1}{\zeta} Z'_{\frac{1}{n+2}}(\zeta) + \left(1 - \frac{1}{(n+2)^2 \zeta^2} \right) Z_{\frac{1}{n+2}}(\zeta) = 0, \quad (36)$$

where $\zeta = \frac{2\sqrt{\beta}}{n+2} x^{\frac{n+2}{2}}$. Equation (36) is the Bessel differential equation whose solutions are the family of Bessel functions of order $\frac{1}{n+2}$.

Equation (34) has been studied in detail by Swanson and Headley [30], who defined its solutions as $A_n(x)$ and $B_n(x)$. It is clear that when $n = 1$ the functions $A_n(x)$ and $B_n(x)$ become the standard Airy functions $\text{Ai}(x)$ and $\text{Bi}(x)$, respectively. An important remark is that they found that the general behavior of the solutions is different depending on the parity of n . This is something to be expected since the branches of the power-law parabolas can have different signs depending on which side of the origin they extend to.

We can now refer to Eq. (34) as the Airy differential equation of order n . In other words, its solutions can be seen as higher-order Airy functions, and similar to the Airy functions, they can also be cast in terms of Bessel functions of fractional order. It should therefore be clear that given adequate conditions in a power-law gradient-index medium, it is then possible to obtain higher-order Airy beams.

We want to remark that the study presented in this section can also be applied to investigate the homologous problem in

quantum mechanics for the Schrödinger equation of a particle confined within an infinite wall and a power-law potential $V(x) = \beta x^n$, with β being constant [31–35]. For the particular case of the linear potential [36], the standing Airy wave packet arises from considering a particle subject to a constant force, e.g., gravitational force, so that when it takes the value of zero, the particle wave packet is still described by the Airy function but is accelerating away from the infinite wall as a consequence of removing the stabilizing force [26].

VI. CONCLUSIONS

In this paper we introduced the physical principles that govern the existence and propagation of Airy beams. We have shown that the nondiffracting characteristics of Airy beams can be explained under the formalism of traveling Hankel waves originally introduced to describe Bessel beams. This is possible because Bessel and Airy functions are intimately related to each other, with the latter being the Bessel functions of fractional order equal to $\frac{1}{3}$. We introduced the two Hankel components of the Airy beam, namely, $\text{Ai}H^{(1)}(\cdot)$ and $\text{Ai}H^{(2)}(\cdot)$ and showed that the latter bears the parabolic caustic property of the beam. It was shown that the superposition of these Hankel components fully explains in simple and straightforward terms propagation characteristics of the Airy beam, such as self-healing and double focusing. Also, this approach allowed us to establish the needed expression to compute the maximum propagation distance of finite-energy Airy beams. We addressed the physical constraints of why an “ideal” Airy beam of infinite extent cannot exist and provided a quantitative method to obtain the maximum extent of an Airy beam of a given wavelength. Finally, by studying the solution of the paraxial wave equation in power-law gradient-index media we demonstrated the possibility of creating higher-order Airy beams.

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