Ermakov–Lewis symmetry in photonic lattices

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We present a class of waveguide arrays that is the classical analog of a quantum harmonic oscillator, where the mass and frequency depend on the propagation distance. In these photonic lattices, refractive indices and second-neighbor couplings define the mass and frequency of the analog quantum oscillator, while first-neighbor couplings are a free parameter to adjust the model. The quantum model conserves the Ermakov–Lewis invariant, thus the photonic crystal also possesses this symmetry.   © 2014 Optical Society of America

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Symmetry in classical and quantum mechanics provides a tool to gain insight from complex systems [1–3]. In their purest form, abstract invariants are related to physical conservation laws, as found by Noether [4]. In practice, the sets of transformations related to symmetry help us make tractable an otherwise complex-in-appearance model [1]. Here, we are interested in such an optical system: a specific class of photonic lattices [5] composed of waveguides that interact with their first- and second-nearest neighbors through evanescent coupling. Further, the refractive indices and the separation between waveguides are functions of the propagation distance. In this Letter, we will show that such a class of photonic lattices is the classical analog of a propagation-dependent harmonic oscillator with an Ermakov–Lewis invariant (ELI) [6–8]. This relation to the time-dependent harmonic oscillator makes these photonic crystals suitable candidates for the classical simulation of diverse quantum problems: e.g., a particle moving in a magnetic field or under the presence of friction. Finally, we will provide an analytic propagator for classical light impinging on these arrays of coupled waveguides through the symmetry transformations related to the ELI.

Our photonic lattice is semi-infinite and composed of individual waveguides whose refractive indices vary linearly with their position in the array times a common function of the propagation distance. The couplings between first- and second-neighbor waveguides vary by the square root of a function of the position in the array times a common function of the propagation distance. There is one particular function for first neighbors and another for second neighbors. In short, it is described by the differential set

$$i\partial_z \mathcal{E}_n(z) + \alpha_0(z) n \mathcal{E}_n(z)$$
$$+ \alpha_1(z)[f_{n+1}\mathcal{E}_{n+1}(z) + f_n \mathcal{E}_{n-1}(z)]$$
$$+ \alpha_2(z)[g_{n+2}\mathcal{E}_{n+2}(z) + g_n \mathcal{E}_{n-2}(z)] = 0, \quad (1)$$

with $f_n = \sqrt{n}$, $g_n = \sqrt{n(n-1)}$, and $\mathcal{E}_{-|n|}(z) = 0$, where the field amplitude at the nth waveguide is $\mathcal{E}_n(z)$, the common z-dependent refractive index is given by $\alpha_0(z)$, and the first- and second-neighbor coupling functions by $\alpha_1(z)$ and $\alpha_2(z)$, respectively. The corresponding

array of waveguides would not necessarily be one dimensional, in order to fulfill the relations for first- and second-neighbor couplings. If we define a state vector, $|\psi(z)\rangle = \sum_{j=0}^{\infty} \mathcal{E}_j(z)|j\rangle$, via the field amplitudes at the jth waveguides $\mathcal{E}_j$, the differential set can be written as a Schrödinger-like equation,

$$i\partial_z |\psi(z)\rangle = \hat{H}|\psi(z)\rangle, \quad (2)$$

in terms of creation (annihilation) operators $\hat{a}^\dagger$ ($\hat{a}$):

$$\hat{H} = -[\alpha_0(z)\hat{n} + \alpha_1(z)(\hat{a} + \hat{a}^\dagger) + \alpha_2(z)(\hat{a}^2 + \hat{a}^{\dagger 2})]. \quad (3)$$

The action of the bosonic creation (annihilation) and the number of operators are given by $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ ($\hat{a}|n\rangle = \sqrt{n}|n\rangle$)) and $\hat{n}|n\rangle = n|n\rangle$, in that order. At this point we can move to a normalized canonical position and momentum, $\hat{a} = (1/\sqrt{2})(\hat{q} + i\hat{p})$ and $\hat{a}^\dagger = (1/\sqrt{2})(\hat{q} - i\hat{p})$, to rewrite the Hamiltonian as

$$\hat{H} = -\left[\frac{1}{2M(z)}\hat{p}^2 + \frac{1}{2}M(z)\Omega^2(z)\hat{q}^2 + \sqrt{2}\alpha_1(z)\hat{q} - \frac{\alpha_0(z)}{2}\right], \quad (4)$$

where the z-dependent mass and frequency are

$$M(z) = \frac{1}{\alpha_0(z) - 2\alpha_2(z)}, \quad (5)$$

$$\Omega^2(z) = \alpha_0^2(z) - 4\alpha_2^2(z). \quad (6)$$

Note that we have the restriction $\alpha_0(z) \neq 2\alpha_2(z)$, between the z-dependent refractive index and second-neighbor couplings, in order to work with a well-behaved mass function. In this form, a displacement and overall phase help us to simplify the dynamics:

$$|\psi(z)\rangle = e^{-i\int\varphi(z)dz}e^{-i[u(z)\hat{p}+M(z)\dot{u}(z)\hat{q}]}|\xi(z)\rangle, \quad (7)$$

with $\varphi(z) = [\alpha_0(z) + M(z)\dot{u}^2(z) + M(z)\Omega^2(z)u^2(z)]/2$ and the auxiliary function $u(z)$ such that it fulfills $\ddot{u}(z) + \dot{M}(z)\dot{u}(z)/M(z) + \Omega^2(z)u + \sqrt{2}\alpha_1(z)/M(z) = 0$,

where we have used the shorthand notation $\dot{u}(z) = \partial_z u(z)$. This is a good point to stop and note that the first-coupling neighbor $\alpha_1(z)$ influences the propagation in our photonic lattices, only through its role in defining the auxiliary function, $u(z)$. Thus, we can rewrite Eq. (2) as a harmonic oscillator, where the mass and frequency depend on the propagation distance,

$$i\partial_t|\xi(t)\rangle = \left[\frac{1}{2m(t)}\hat{p}^2 + \frac{1}{2}m(t)\omega^2(t)\hat{q}^2\right]|\xi(t)\rangle, \quad (8)$$

and we have made the variable change $t = -z$, leading to $m(t) = M(-t)$ and $\omega(t) = \Omega(-t)$, for the sake of simplicity. Such a quantum model shows an ELI [7–12]:

$$\hat{I} = \frac{1}{2}\left\{\left[\frac{\hat{q}}{\rho(t)}\right]^2 + [\rho(t)\hat{p} - m(t)\dot{\rho}(t)\hat{q}]^2\right\}, \quad (9)$$

where the new auxiliary function fulfills the Ermakov equation [6]:

$$\ddot{\rho}(t) + \frac{\dot{m}(t)}{m(t)}\dot{\rho}(t) + \omega^2(t)\rho(t) = \frac{1}{m^2(t)\rho^3(t)}. \quad (10)$$

The ELI possesses a related set of symmetry transformations [12,13]:

$$|\xi(t)\rangle = e^{i\frac{m(t)\dot{\rho}(t)}{2\rho(t)}\hat{q}^2}e^{-i\frac{\ln\rho(t)}{2}(\hat{p}\hat{q}+\hat{q}\hat{p})}|\zeta(t)\rangle, \quad (11)$$

which are equivalent to the displacement: the first exponential in the right-hand side (rhs) term; and squeezing operations: the second exponential in the rhs term. This squeezed and displaced basis diagonalizes our system:

$$i\partial_t|\zeta(t)\rangle = \frac{1}{2m(t)\rho^2(t)}(\hat{p}^2 + \hat{q}^2)|\zeta(t)\rangle, \quad (12)$$

$$= \frac{1}{m(t)\rho^2(t)}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|\zeta(t)\rangle. \quad (13)$$

It is straightforward to solve this Schrödinger-like equation, as all dependence on the propagation has been factorized to a common term. Thus, the analytic propagator for any given initial state is

$$|\zeta(t)\rangle = e^{-i\int\frac{1}{m(t)\rho^2(t)}dt(\hat{a}^\dagger\hat{a}+\frac{1}{2})}|\zeta(0)\rangle, \quad (14)$$

where the state vector $|\zeta(0)\rangle$ holds the information from the initial field amplitudes that impinge upon the waveguide array. These transformations and evolution in the new frame of reference are enough to provide an impulse function in the original frame. Notice that light impinging on just the jth waveguide, in the original frame of reference, will be equivalent to having the initial wave function, $(1/\sqrt{2^j j!})(1/\pi)^{1/4}e^{-q^2/2}H_j(q)$, where $H_n(x)$ is the nth Hermite polynomial [14]. Thus, a beam of light impinging on just the first waveguide of the array is equivalent to an initial Gaussian wave function in dimensionless canonical space, $\Psi_0(q) = e^{-q^2/2}/\pi^{1/4}$. This has to be taken into account when going to or from the transformed frame.

Further, as our photonic lattice with first- and second-neighbor coupling is equivalent to a z-dependent harmonic oscillator, Eq. (4), it can be used to simulate a charged particle moving nonrelativistically in the presence of a magnetic field [7,9], if the parameters are adjusted such that the mass is constant, $m(t) = m$, and the frequency is proportional to the amplitude of the magnetic field, $\omega(t) = B(t)/2$. It can also be used to simulate quantum oscillators in the presence of friction, if the mass is written as $m(t) = m/F(t)$ [15]. Setting $F(t) = e^{-\gamma t}$ with constants γ, and $\omega(t)$, leads to the Caldirola–Kanai Hamiltonian [16,17]. Analytical closed forms for the auxiliary function $\rho(t)$ can be obtained for a variety of driving frequencies: e.g., $\omega(t) = c$ leads to $\rho^2(t) = \omega(t)^{-1}$, $\omega(t) = ct^{-1}$ gives $\rho^2(t) = t(c^2 - 1/4)^{-1/2}$, and $\omega(t) = ct^k$ yields $\rho^2(t) = \pi(k + 1)t[J_{((k+1)/2)^2}(k + 1c^2t^{k+1}) + Y_{((k+1)/2)^2}(k + 1c^2t^{k+1})]/2$. In all cases, c is a constant [8,10]. Of course, each case provides its own set of design challenges in order to bring the photonic analogs to the laboratory.

Let us consider a simple example by setting a harmonic oscillator with constant mass and frequency, $M(z) = \Omega(z) = 1$. This case is related to an array with parameters $\alpha_0(z) = 1$, $\alpha_1(z) = 1$, and $\alpha_2(z) = 0$; i.e., there are only first-neighbor couplings in the photonic lattice. This realization of our lattice is equivalent up to a constant with the Glauber–Fock oscillator lattice, which allows for Bloch-like revivals, and has been produced experimentally [18]. Figure 1 shows the propagation of light intensity in such a lattice, for an initial field impinging on just the $j = 10$ waveguide. Increasing the constant-driving frequency in the quantum model is equivalent to including second-neighbor couplings in the optical model. As the quantum models are the same, the dynamics of light propagating through the new waveguide array are similar to that of the Glauber–Fock oscillator lattice; e.g., the spatial frequency of the Bloch-like revivals becomes higher in the lattice, including second-neighbor couplings, compared to just the first-neighbor couplings. For example, Fig. 2 shows propagation in our lattice with parameters, $\alpha_0(z) = 5/2$, $\alpha_1(z) = 1$, and $\alpha_2(z) = 3/4$, which are related to the oscillator parameters, $M(z) = 1$ and $\Omega(z) = 2$.
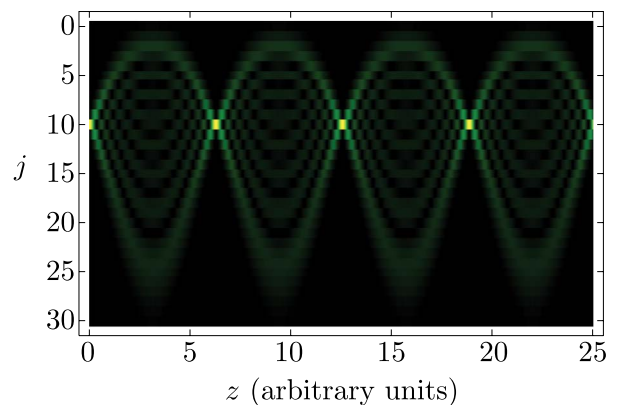

Fig. 1.   Light intensity propagation for a beam impinging on the $j = 10$ waveguide of a lattice with parameters, $\alpha_0(z) = \alpha_1(z) = 1$, and $\alpha_2(z) = 0$; this is equivalent to an oscillator with constant mass and frequency, $M(z) = \Omega(z) = 1$.
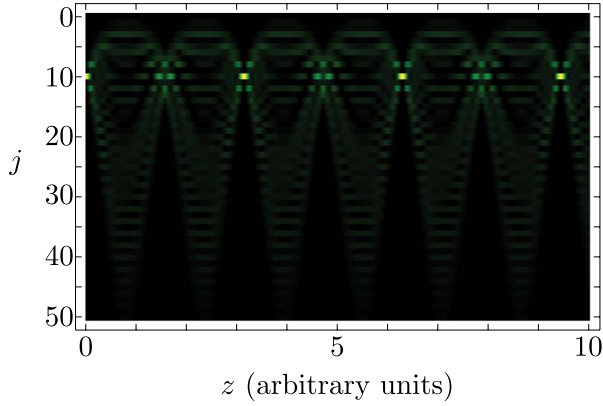
Fig. 2. Light intensity propagation for a beam impinging on the $j = 10$ waveguide of a lattice with parameters $\alpha_0(z) = 5/2$, $\alpha_1(z) = 1$, and $\alpha_2(z) = 3/4$; this is equivalent to an oscillator with constant mass and frequency, $M(z) = 1$ and $\Omega(z) = 2$.
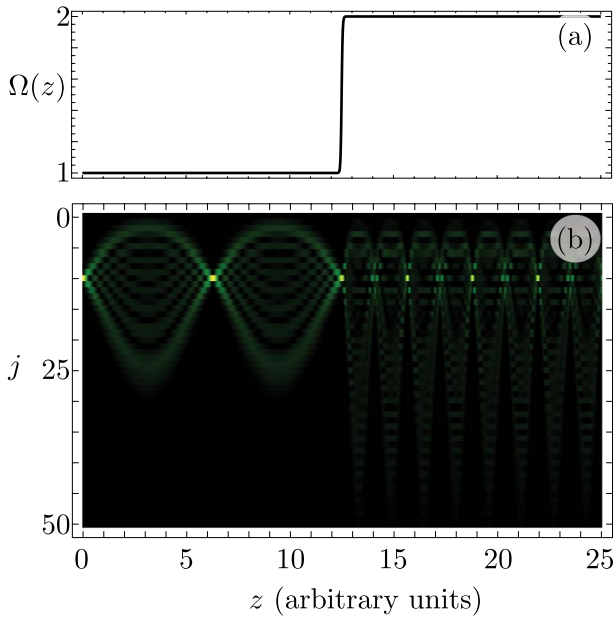

Fig. 3. (a) Profile of the driving frequency, $\Omega(z)$. (b) Light intensity propagation for a beam impinging on the $j = 10$ waveguide of a lattice with parameters $\alpha_0(z) = [M^2(z)\Omega(z)^2 + 1]/[2M(z)]$, $\alpha_1(z) = 1$, and $\alpha_2(z) = [M^2(z)\Omega(z)^2 - 1]/[4M(z)]$; with harmonic oscillator parameters $M(z) = 1$ and $\Omega(z) = [3 + \tanh 20(z - 12.5)]/2$.

We can also study an example bridging the two arrays of waveguides discussed above, by considering a driving frequency, $\Omega(z) = [3 + \tanh \varepsilon(z - z_s)]/2$, which is the equivalent of a smooth, well-behaved step function, where the parameter ε controls the steepness of the jump, and z_s is the switching position, as shown in Fig. 3(a). Thus, for propagation distances $z < z_s$ we

expect coherent oscillations related to the Glauber–Fock oscillator lattice with a given spatial frequency; and then, after switching to distance z_s, we expect oscillations related to the higher spatial frequency of a Glauber–Fock oscillator, including second-neighbor couplings. Figure 3(b) shows the numerical propagation of such a system. This case admits an approximate auxiliary function, $\rho^2(t) = 1 + 1/\omega^2(t) + (1 - 1/\omega^2(t)) \cos \int_{t_s}^{t} 2\omega(\zeta)\mathrm{d}\zeta$ [19,20]. As mentioned before, this is equivalent to a Glauber–Fock oscillator lattice that smoothly transitions from just first-neighbor couplings to first- and second-neighbor couplings. In numerical simulations carried out with lattices consisting of 500 waveguides, the light intensity at the last waveguide was never larger than 10^{-6}.

In short, we have shown that certain nontrivial arrays of photonic waveguides, where refractive indices, as well as first- and second-neighbor couplings, depend on the propagation distance, and can classically simulate the dynamics of a quantum harmonic oscillator with non-constant mass and frequency. The quantum system shows an ELI that defines the underlying symmetry of the photonic crystal and is related to diverse quantum mechanical problems: e.g., a charged particle in the presence of a magnetic field and driving/friction in quantum oscillators.

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