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Relation between the Glauber–Sudarshan and Kirkwood–Rihaczek distribution functions

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The Kirkwood–Rihaczek quasiprobability distribution is written as a vacuum state expectation value of squeeze-like operators and the density matrix. We do this, by writing the position eigenstates as a *squeezing*-like action on the vacuum. This allows us to give a relation between the Glauber–Sudarshan and Kirkwood–Rihaczek quasiprobability distributions.

Keywords: quasiprobability distribution functions; non-classical states

1. Introduction

One of the main aims in the field of quantum optics is the production of non-classical states (NCS), for instance in trapped ions and quantized electromagnetic fields. Such states have been produced recently in experiments around the world, in particular by Wineland's and Haroche's groups [1–5]. However, NCS being quantum mechanical objects, besides being fragile because of the influence of the environment, once a given nonclassical state has been produced, it is important to count with mechanisms that allow their measurement, a key problem in quantum mechanics.

Quasiprobability distribution functions (QDFs) are widely used in quantum mechanics [6,7] and optical physics [8]. They are of great help in the visualization of such non-classical states. For instance, one can see the compression in phase space when squeezed states of the harmonic oscillator [9,10] (quantized field or an ion oscillating in a trap) are generated, or the negativity of the Wigner function [11] for Fock states [12,13].

Therefore, via QDF it is possible to obtain information from a system by measuring, not only some of its observables, but directly the density matrix, as it is possible to obtain a quasiprobability function from a density matrix or vice versa. One of the possible ways of obtaining such information is by expressing an s-parametrized QDF in its series representation [14]

$$F(\alpha, s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left(\frac{s+1}{s-1}\right)^k \langle \alpha, k|\rho|\alpha, k\rangle \qquad (1)$$

with s the quasiprobability function's parameter that indicates which is the relevant distribution ($s = -1$ Husimi [15], $s = 0$ Wigner [6,11] and $s = 1$ Glauber–Sudarshan [16,17] distribution function), ρ is the density matrix and the states $|\alpha, k\rangle$ are displaced number states [18–20], that may be experimentally produced [21].

Moreover, besides applications in classical optics [22], it has been shown that these phase space distributions can be expressed, in thermofield dynamics, as overlaps between the state of the system and *thermal* coherent states [23], which is probably the reason by which systems subject to decay, may still be 'measured' [24,25].

Wineland's [12] and Haroche's [13] groups used the above expression to measure the Wigner function ($s = 0$ case) of the quantized motion of an ion and a quantized cavity field, respectively. There is somehow already a recipe in the above equation to obtain a quasiprobability distribution function from experimental data: let us write Equation (1) as

$$F(\alpha, s) = \frac{2}{\pi(1-s)} \sum_{k=0}^{\infty} \left(\frac{s+1}{s-1}\right)^k \langle k|D^\dagger(\alpha)\rho D(\alpha)|k\rangle, \qquad (2)$$

where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, with a and $a^\dagger$ the annihilation and creation operators, respectively, is the Glauber displacement operator. Note that, in order to obtain a quasiprobability distribution function we need to do the following: displace the system by an amplitude α and then measure the diagonal elements of the displaced density matrix. These applications in the reconstruction of signals [8] in the classical regime and reconstruction of quantum states of different systems such as ions [12,26] or quantized fields [13] in the quantum regime is a fundamental property that makes them an invaluable tool as a measuring device.

However, as it may not be always possible to reconstruct the Wigner function due, for instance, to the interaction of

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the system with its environment [24,25], then one has to rely on s-parametrized QDFs for the measurement of quantum states. It is of importance to look for new elements that could contribute to the measurement of a wavefunction when the above mentioned mechanisms may not be used. Operators used in the reconstruction process are always Hermitian, therefore real values are always handled, producing real QDFs. However, there is a class of QDFs that is complex and still has the same amount of information as the real Wigner, Glauber–Sudarshan or Husimi distribution functions, namely the Kirkwood–Rihaczek function [27–30]. Along the lines of using QDF to perform the wavefunction measurement, let us think of a hypothetical, but plausible scenario where measurements of observables, position, momentum, let us say, do not guide us to the reconstruction of a QDF. However, via their manipulation, addition or subtraction, it is posible to 'measure' non-Hemitian operators such as creation or annihilation operators. Indeed the result of such manipulation would derive in complex values as the operator measured is non-observable and therefore non-Hermitian. Then connections for reconstruction of complex QDFs could be the answer. In this contribution we would like to re-introduce this class of complex quasiprobability distribution functions and use their relation to the Wigner function to show how it may be related to the Glauber–Sudarshan P-function. This will be possible as we express the Kirkwood–Rihaczek function as an expectation value in terms of the vacuum.

2. Cohen-class distribution functions

A function of the Cohen class is described by the general formula [31]

$$W_C = \frac{1}{2\pi}\iiint \phi(y+\tfrac{1}{2}x')\phi(y-\tfrac{1}{2}x')k(x,u,x',u') \times \exp[-\mathrm{i}(ux'-u'x+u'y)]\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}u' \tag{3}$$

and the choice of the kernel $k(x,u,x',u')$ selects one particular function of the Cohen class. The Wigner function, for instance arises for $k(x,u,x',u')=1$, whereas the ambiguity function is obtained for $k(x,u,x',u')2\pi\delta(x-x')\delta(u-u')$.

2.1. *Wigner function*

Probably the best known QDF is the Wigner function. It may be written in two forms: series representation (see for instance [14]), and from (3), as the integral representation

$$W(q,p) = \frac{1}{2\pi}\int \mathrm{d}u\,\exp(\mathrm{i}up)\left\langle q+\frac{u}{2}\right|\rho\left|q-\frac{u}{2}\right\rangle. \tag{4}$$

Wigner introduced this function $W(q,p)$, known now as his distribution function [6,11] which contains complete information about the state of the system as the density matrix for a pure state is given by $\rho = |\psi\rangle\langle\psi|$.

The Wigner function may be written also, in terms of the (double) Fourier transform of the characteristic function, as

$$W(\alpha) = \frac{1}{4\pi^2}\int \exp(\alpha\beta^* - \alpha^*\beta)C(\beta)\,\mathrm{d}^2\beta, \tag{5}$$

with $\alpha = (q+\mathrm{i}p)/2^{1/2}$ and where $C(\beta)$ in terms of annihilation and creation operators is given by

$$C(\beta) = \mathrm{Tr}\,\{\rho\exp(\beta a^\dagger - \beta^* a)\}, \tag{6}$$

also known as the ambiguity function in classical optics [32].

3. The Kirkwood–Rihaczek quasidistribution function

Now we turn our attention to a lesser known distribution, the Kirkwood–Rihaczek function, which may be written using the notation above as [33]

$$K(\beta) = \int \mathrm{d}^2\alpha\,\exp(\beta\alpha^* - \beta^*\alpha)\exp\left(\frac{\alpha^2-\alpha^{*2}}{4}\right)C(\alpha), \tag{7}$$

and may also be expressed as the double Fourier transform

$$K(X,Y) = \int \mathrm{d}u\,\mathrm{d}v\,\exp(-\mathrm{i}uY)\exp(\mathrm{i}vX)\mathrm{Tr} \times\{\rho\exp(\mathrm{i}v\hat{q})\exp(\mathrm{i}u\hat{p})\}, \tag{8}$$

where we have defined $\beta = (X+\mathrm{i}Y)/2^{1/2}$ and for the trace, that may be taken in many forms, we use

$$\mathrm{Tr}\,\{\rho A\} = \langle\psi|A|\psi\rangle = \int_{-\infty}^{\infty}\mathrm{d}q\,\langle_p q|\psi\rangle\langle\psi|A|q\rangle_p, \tag{9}$$

where we have added the subscript p to emphasize the use of position eigenstates.

We will now do an analysis similar to the one done in [14]. We relate the Kirkwood–Rihaczek function to the Wigner function by using (7), via the following exponential of derivatives

$$K(\beta) = \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}\right)\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)W(\beta). \tag{10}$$

In the above equation we will use a non-integral expression for the Wigner function [14]

$$W(\beta) = \mathrm{Tr}\left[(-1)^{a^\dagger a}D^\dagger(\beta)\rho D(\beta)\right]. \tag{11}$$

Rearranging the displacement operators and the parity operator, we obtain

$$W(\beta) = \mathrm{Tr}\left[(-1)^{a^\dagger a}\rho D(2\beta)\right], \tag{12}$$

where we have used the trace property $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$ and the following identity $(-1)^{a^\dagger a}D^\dagger(\beta) = D(\beta)(-1)^{a^\dagger a}$.

Now we use the factorized form of the Glauber displacement operator [16] $D(2\beta) = e^{-2|\beta|^2}\exp(2\beta a^\dagger)\exp(-2\beta^* a)$ to obtain

$$W(\beta) = \mathrm{Tr}\left[(-1)^{a^\dagger a}\rho\exp(-2|\beta|^2) \times\exp(2\beta a^\dagger)\exp(-2\beta^* a)\right]. \tag{13}$$

Therefore, we have that the Kirkwood–Rihaczek function may be written as

$$K\left(\beta,\beta^*\right)=\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}\right)\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)W\left(\beta,\beta^*\right)$$
$$=\mathrm{Tr}\left[(-1)^{a^\dagger a}\rho\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}\right)\times\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)D\left(2\beta\right)\right]. \quad (14)$$

The calculation of the exponential of derivatives of the Glauber operator will be tedious but straightforward. We will follow [34] for the calculations

$$\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)D\left(2\beta\right)=\exp(-\beta^2)\exp\left[2\beta\left(a^\dagger+a-\beta^*\right)\right]\times\exp(a^2)\exp(-2\beta^*a). \quad (15)$$

By using the expression for the generating function for Hermite polynomials [35]

$$\exp(-t^2+2tx)=\sum_{k=0}^{\infty}H_k\left(x\right)\frac{t^k}{k!} \quad (16)$$

we can express the above equation as

$$\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)D\left(2\beta\right)=\sum_{k=0}^{\infty}H_k\left(a^\dagger+a-\beta^*\right)\frac{\beta^k}{k!}\times\exp(a^2)\exp(-2\beta^*a). \quad (17)$$

From the above equation, it is easy to note that

$$\frac{\partial^{2n}}{\partial\beta^{2n}}\sum_{k=0}^{\infty}H_k\left(x\right)\frac{\beta^k}{k!}=\sum_{k=0}^{\infty}H_{k+2n}\left(x\right)\frac{\beta^k}{k!} \quad (18)$$

such that

$$\exp\left(-\frac{1}{4}\frac{\partial^2}{\partial^2\beta}\right)\exp\left(\frac{1}{4}\frac{\partial^2}{\partial^2\beta^*}\right)D\left(2\beta\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^n\left(\beta\right)^k}{n!k!}H_{k+2n}\left(a^\dagger+a-\beta^*\right)\times\exp(a^2-2\beta^*a). \quad (19)$$

Now we use the integral form of the Hermite polynomials [35]

$$H_p\left(x\right)=\frac{2^p}{\pi^{1/2}}\int_{-\infty}^{\infty}\left(x+\mathrm{i}t\right)^p\exp(-t^2)\,\mathrm{d}t \quad (20)$$

to obtain

$$K\left(\beta,\beta^*\right)=\frac{\exp(-\beta^{*2})\exp(-2\beta\beta^*)}{\pi^{1/2}}\times\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t\exp\left[\left(-2(2^{1/2})x+2\beta^*+2\beta\right)\mathrm{i}t\right]\times\exp(-2x^2)\exp\left[2(2^{1/2})x\left(\beta^*+\beta\right)\right]\times{}_p\left\langle x\right|\exp(a^2)\exp(-2\beta^*a)\left(-1\right)^{a^\dagger a}\rho\left|x\right\rangle_p \quad (21)$$

by using

$$\int_{-\infty}^{\infty}\exp(-\mathrm{i}yt)\,\mathrm{d}t=2\pi\delta\left(y\right). \quad (22)$$

By setting $y=2\sqrt{2}x-2\beta^*-2\beta$ we have

$$K\left(\beta,\beta^*\right)=2\pi^{1/2}\exp(-\beta^{*2})\exp(-2\beta\beta^*)\times\int_{-\infty}^{\infty}\mathrm{d}x\delta\left(2(2^{1/2})x-2\beta^*-2\beta\right)\times\exp(-2x^2)\exp\left[2(2^{1/2})x\left(\beta^*+\beta\right)\right]\times\left\langle{}_px\right|\exp(a^2)\exp(-2\beta^*a)\left(-1\right)^{a^\dagger a}\rho\left|x\right\rangle_p. \quad (23)$$

Making use of the identity $\delta\left(\alpha x\right)=\delta\left(x\right)/|\alpha|$ we finally obtain

$$K\left(\beta,\beta^*\right)=\left(\frac{\pi}{2}\right)^{1/2}\exp(\beta^2-\beta^{*2})\times\left\langle{}_pX\right|\exp\left[\left(a-\beta^*\right)^2\right]\left(-1\right)^{a^\dagger a}\rho\left|X\right\rangle_p. \quad (24)$$

The position eigenstate $|X\rangle_p$ may be written as

$$|X\rangle_p=\sum_{n=0}^{\infty}|n\rangle\langle n\left|X\right\rangle_p \quad (25)$$

or

$$|X\rangle_p=\sum_{n=0}^{\infty}\psi_n(X)|n\rangle \quad (26)$$

with

$$\psi_n(X)=\frac{\exp(-X^2/2)H_n(X)}{\left(2^n\pi^{1/2}n!\right)^{1/2}}$$

such that the position eigenstate may we re-written as

$$|X\rangle_p=\frac{\exp(-X^2/2)}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{H_n(X)}{2^{n/2}n!}a^{\dagger n}|0\rangle, \quad (27)$$

which may be added via the generating function for Hermite polynomials (16) to give

$$|X\rangle_p = \frac{\exp(-X^2/2)}{\pi^{1/4}} \exp\left(-\frac{a^{\dagger 2}}{2} + 2^{1/2} a^\dagger X\right) |0\rangle. \quad (28)$$

In the above equation the application of an operator to the vacuum produces the position eigenstate.[1]

By using this expression in (24) we can write

$$\begin{aligned} K\left(\beta, \beta^*\right) = & \frac{\exp(\beta^2 - X^2)}{2^{1/2}} \\ & \times \langle 0| \exp\left(\frac{a^2}{2} - 2^{1/2} \mathrm{i} Y a\right) (-1)^{a^\dagger a} \\ & \times \rho \exp\left(-\frac{a^{\dagger 2}}{2} + 2^{1/2} a^\dagger X\right) |0\rangle, \quad (29) \end{aligned}$$

or

$$\begin{aligned} K\left(\beta, \beta^*\right) = & \frac{\exp(\beta^2 - X^2)}{2^{1/2}} \\ & \times \langle 0| \exp\left(\frac{a^2}{2} + 2^{1/2} \mathrm{i} Y a\right) \\ & \times \rho \exp\left(-\frac{a^{\dagger 2}}{2} + 2^{1/2} a^\dagger X\right) |0\rangle, \quad (30) \end{aligned}$$

that may be finally written in terms of coherent states

$$\begin{aligned} K(\beta, \beta^*) = & \frac{\exp(\beta^2 + Y^2)}{2^{1/2}} \langle -2^{1/2} \mathrm{i} Y| \exp\left(\frac{a^2}{2}\right) \rho \\ & \times \exp\left(-\frac{a^{\dagger 2}}{2}\right) |2^{1/2} X\rangle. \quad (31) \end{aligned}$$

We can relate the Kirkwood function to the Glauber–Sudarshan P-function [16,17] by using the relation $\rho = \int \mathrm{d}^2\alpha P(\alpha) |\alpha\rangle\langle\alpha|$, i.e.

$$\begin{aligned} K(\beta, \beta^*) = & \frac{\exp(\beta^2 + Y^2)}{2^{1/2}} \int \mathrm{d}^2\alpha P(\alpha) \exp\left(\frac{\alpha^2 - \alpha^{*2}}{2}\right) \\ & \times \langle -2^{1/2} \mathrm{i} Y|\alpha\rangle \langle 2^{1/2} X|\alpha\rangle, \quad (32) \end{aligned}$$

or

$$\begin{aligned} & K(\beta, \beta^*) \\ & = \frac{\exp(\mathrm{i}XY)}{2^{1/2}} \int \mathrm{d}^2\alpha P(\alpha) \exp\left(\frac{\alpha^2 - \alpha^{*2}}{2} - |\alpha|^2\right) \\ & \times \exp\left[2^{1/2}(X\alpha^* - \mathrm{i}Y\alpha)\right]. \quad (33) \end{aligned}$$

Therefore we have written the Kirkwood–Rihaczek function as an expectation value in terms of the vacuum state, just as the Q-function may be written as a coherent states expectation value, the Wigner and Glauber–Sudarshan functions in terms of a series of displaced number states expectation values [14], and relate it to the Glauber–Sudarshan P-function.

4. Conclusions

We have written the position eigenstates as a 'displacement' or 'squeezing' of the vacuum state. In fact it is a non-unitary squeeze-like operator applied on the vacuum, which is not surprising as position eigenstates are not normalizable. This has allowed us to use a former expression for the Kirkwood–Rihaczek distribution function to write it as an expectation value in terms of the vacuum state. This made it easy to relate this function to the Glauber–Sudarshan P-function [36].

Note

1. Note that one can write exponentials of annihilation operators just before the vacuum.

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