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Mirror-field-atom interaction: Hamiltonian diagonalization

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We show that the interaction between a movable mirror with a quantized field that interacts with a two-level atom may be simplified via a transformation that involves Susskind-Glogower operators (SGO). By using this transformation it is easy to show that we can cast the Hamiltonian, after a series of small rotations, into an effective Hamiltonian that may be solved. We would like to stress that the transformation in terms of SGO already simplifies enough the Hamiltonian in the sense that, in an exact way, it "eliminates" one of the three-subsystems, namely the quantized field.

Keywords: Atom-field interaction; mirror field interaction; non-classical states.

Mostramos como la interacción entre un espejo móvil y un campo cuantizado el cual a su vez interactúa con un átomo de dos niveles puede ser simplificada vía el uso de operadores de Susskind-Glogower (OSG). Usando una transformación formada con estos operadores, es fácil de mostrar que el hamiltoniano se puede llevar, después de una serie de pequeñas rotaciones, a un hamiltoniano efectivo el cual es soluble. Nos gustaría enfatizar que el uso de la transformación en términos de los OSG permite ya simplificar bastante el hamiltoniano en el sentido de que, en forma exacta, "elimina" unos de los tres sub-sistemas, en este caso, el campo electromagnético.

Descriptores: Interacciones átomo-campo; campo-espejo móvil; estados no-clásicos.

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1. Introduction

Recently, special attention has been devoted to a system consisting of a cavity field and a movable mirror [1, 2]. This is due to the fact that for such a system we can produce non-classical states [3], particularly the macroscopic superposition of at least two coherent states, *i.e.* Schrödinger cat-states. The concept of superposition of states plays a fundamental role in understanding the foundations of quantum mechanics, this is why the generation of non-classical states, such as squeezed states [4], and the particularly important limit of extreme squeezing, *i.e.* Fock or number states [5], has been widely studied in several systems. It is known that a non-linear interaction can generate Schrödinger cat-states. The non-linear interaction used to generate such states is the one produced by a Kerr medium [6, 7] which corresponds to a quadratic Hamiltonian in the number field operator [8, 9]. Our main motivation to make the field-mirror system interact with an atom is to look for the possibility to extract information about it by later measuring atomic properties. This, because it is well known that several quasiprobability reconstruction techniques [10] for the quantized field [11] or the vibrational motion of an ion [12], rely on the measurement of atomic states. Therefore, the passage of atoms through such systems, could give us information, not only about the states of the mirror or field, but about its dynamical interaction. This is, the passage of a two-level atom through a cavity containing a movable mirror may give us information about the entanglement between mirror and field. The purpose of this contribution is not to study this possibility, however, but to show how the total system may be simplified, by several rotations (some of them small rotations, *i.e.* approximations) that diagonalize the Hamiltonian in such a way that a solution may be easily obtained. We will study the possibility of reconstructing the mirror-field interaction elsewhere.

2. Interaction between the cavity and the mirror

The interaction between an electromagnetic field and a movable mirror (treated quantum mechanically) has a relevant Hamiltonian given by [8] (we set $\hbar = 1$)

$$H_{f-m} = \omega a^\dagger a + \nu b^\dagger b - g a^\dagger a (b^\dagger + b), \tag{1}$$

where a and $a^\dagger$ are the annihilation and creation operators for the cavity field, respectively. The field frequency is ω. b and $b^\dagger$ are the annihilation and creation operators for the mirror oscillating at a frequency ν and

$$g = \frac{\omega}{L}\sqrt{\frac{\hbar}{2m\nu}}, \tag{2}$$

with L and m the length of the cavity and the mass of the movable mirror.

3. Mirror-Field-Atom interaction

If we pass a two-level atom through a cavity with a movable mirror as the one described by Eq. (1), we have have to add the free Hamiltonian for the atom and the interaction with the quantized field, so we obtain [13]

$$H_{a-f-m} = \frac{\omega_0}{2}\sigma_z + \lambda\left(a\sigma_+ + a^\dagger\sigma_-\right) + \omega a^\dagger a + \nu b^\dagger b - g a^\dagger a(b^\dagger + b), \quad (3)$$

where λ is the atom-field interaction constant, ω_0 is the atomic transition frequency and σ_- (σ_+) is the lowering (raising) operator for the atom, with $[\sigma_+, \sigma_-] = 2\sigma_z$.

We consider the on-resonant interaction between the field an the atom, *i.e.* $\omega = \omega_0$, and pass to an interaction picture, taking advantage that the operator $\omega(a^\dagger a + 2\sigma_z/2)$ commutes with all the other operators involved in the Hamiltonian, to obtain

$$\widehat{H} = \nu\widehat{N} + \chi\widehat{n}\left(b + b^\dagger\right) + \lambda\left(a\sigma_+ + a^\dagger\sigma_-\right). \quad (4)$$

The quantities $\widehat{n} = a^\dagger a$ and $\widehat{N} = b^\dagger b$ are the number operators for the field and mirror, respectively. We will use the Susskind-Glogower operators [14]

$$V = \frac{1}{\sqrt{\hat{n}+1}}a, \qquad V^\dagger = a^\dagger\frac{1}{\sqrt{\hat{n}+1}}, \quad (5)$$

that satisfy the commutation relation $[V, V^\dagger] = |0\rangle\langle 0|$ to transform the above Hamiltonian with the following matrix operator [16]

$$M^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & V^\dagger \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}, \quad (6)$$

such that we can rewrite the interaction Hamiltonian as

$$\widehat{H} = \widehat{H}_V + \widehat{\rho}_{22}^0 \quad (7)$$

where

$$\widehat{H}_V = M^\dagger \begin{pmatrix} \nu\widehat{N}+\chi\widehat{n}\left(b+b^\dagger\right) & \lambda\sqrt{\widehat{n}+1} \\ \lambda\sqrt{\widehat{n}+1} & \nu\widehat{N}+\chi\left(\widehat{n}+1\right)\left(b+b^\dagger\right) \end{pmatrix} M, \quad (8)$$

and

$$\widehat{\rho}_{22}^0 = \begin{pmatrix} 0 & 0 \\ 0 & \nu\widehat{N} \end{pmatrix} . |0\rangle\langle 0|$$

Note that

$$\left[\widehat{H}_V, \rho_{22}^0\right] = 0,$$

therefore we can write the evolution operator as

$$\widehat{U}(t) = e^{-i\widehat{H}t} = e^{-i\widehat{H}_V t} e^{-i\widehat{\rho}_{22}^0 t}. \quad (9)$$

In order to calculate $e^{-i\widehat{H}_V t}$, we develop the exponential in Taylor series take into account that

$$\widehat{H}_V^k = M^\dagger \begin{pmatrix} \nu\widehat{N}+\chi\widehat{n}\left(b+b^\dagger\right) & \lambda\sqrt{\widehat{n}+1} \\ \lambda\sqrt{\widehat{n}+1} & \nu\widehat{N}+\chi\left(\widehat{n}+1\right)\left(b+b^\dagger\right) \end{pmatrix}^k M, \quad k \geq 1, \quad (10)$$

so we write the evolution operator as

$$\widehat{U}(t) = M^\dagger e^{-i\widehat{\widetilde{H}}t} M e^{-i\widehat{\rho}_{22}^0 t} + \left(\frac{1-\sigma_Z}{2}\right)|0\rangle\langle 0| e^{-i\widehat{\rho}_{22}^0 t},$$

where

$$\widehat{\widetilde{H}} = \begin{pmatrix} \nu\widehat{N}+\chi\widehat{n}\left(b+b^\dagger\right) & \lambda\sqrt{\widehat{n}+1} \\ \lambda\sqrt{\widehat{n}+1} & \nu\widehat{N}+\chi\left(\widehat{n}+1\right)\left(b+b^\dagger\right) \end{pmatrix}. \quad (11)$$

Up to here we have realized already a relevant simplification: this Hamiltonian, unlike the one in equation (4), has field operators in it that commute with each other, and, because they commute with the other sub-systems operators, they may be treated from now on as classical numbers. Therefore, we have effectively and exactly eliminated one sub-system, namely the field, from the initial problem.

Now we will take advantage of the difference in order of magnitudes (the atom-field interaction constant is much larger than the mirror-field interaction constant) of the different constants in this interaction to produce an effective Hamiltonian which being diagonal, may be solved exactly. For this we will use a small rotation approach proposed by Klimov and Sánchez-Soto [15]. First, let us do an exact rotation to the Hamiltonian and obtain

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

with

$$H_R = R\widetilde{H}R^\dagger,$$

so that we have

$$\widehat{H}_R = \nu\widehat{N} + \chi\left(\widehat{n} + \frac{1}{2}\right)\left(b + b^\dagger\right) + \lambda\sigma_z\sqrt{\widehat{n}+1} + \frac{\chi}{2}\left(\sigma_+ + \sigma_-\right)\left(b + b^\dagger\right). \quad (12)$$

Now we apply the following small rotations

$$\widehat{U}_1 = e^{\xi_1\left(b^\dagger\sigma_+ - b\sigma_-\right)},$$

$$\widehat{U}_2 = e^{\xi_2\left(b\sigma_+ - b^\dagger\sigma_-\right)},$$

we call the small because we will consider (see below) $\xi_1, \xi_2 \ll 1$, *i.e.* they will perform tiny rotations. Transforming (12) with the above operators we obtain

$$H_2 = U_2 U_1 H_R U_1^\dagger U_2^\dagger.$$

Due to the fact that the ξ 's are small we remain to first order when we develop the exponentials in Taylor series so that we finally obtain

$$
\begin{aligned}
\widehat{H}_2 \approx{}& \nu\widehat{N} + \lambda\sqrt{\widehat{n}+1}\sigma_z + \chi\left(\widehat{n}+\frac{1}{2}\right)\left(b+b^\dagger\right) \\
&+ \left(\xi_1+\xi_2\right)\chi\sigma_z\left(b+b^\dagger\right)^2 \\
&+ \left(\xi_2-\xi_1\right)\left(\frac{\chi}{2}\left(\sigma_+ + \sigma_-\right)^2 + \chi\left(\widehat{n}+\frac{1}{2}\right)\left(\sigma_+ + \sigma_-\right)\right) \\
&+ \left(\frac{\chi}{2} - \xi_1\left(\nu+\lambda\sqrt{\widehat{n}+1}\right)\right)\left(b^\dagger\sigma_+ + b\sigma_-\right) \\
&+ \left(\frac{\chi}{2} + \xi_2\left(\nu-\lambda\sqrt{\widehat{n}+1}\right)\right)\left(b\sigma_+ + b^\dagger\sigma_-\right).
\end{aligned}
$$

We can choose

$$
\xi_1\left(\widehat{n}\right) = \frac{\chi}{2\left(\nu+\lambda\sqrt{\widehat{n}+1}\right)},
$$

$$
\xi_2\left(\widehat{n}\right) = -\frac{\chi}{2\left(\nu-\lambda\sqrt{\widehat{n}+1}\right)},
$$

so that the last two terms of $\widehat{H}_2$ become zero. For the other terms we need to calculate

$$
\xi_1+\xi_2 = \frac{\lambda\chi\sqrt{\widehat{n}+1}}{\lambda^2\left(\widehat{n}+1\right)-\nu^2}, \tag{13}
$$

and using the fact that the ξ's are small we can also neglect the term

$$
\xi_2-\xi_1 = \frac{\chi\nu}{\left(\lambda^2\left(\widehat{n}+1\right)-\nu^2\right)} \sim \frac{1}{\lambda^2} \sim 0. \tag{14}
$$

Taking into account that

$$
\frac{\chi^2}{\lambda^2\left(\widehat{n}+1\right)-\nu^2} \approx \frac{\chi^2}{\lambda^2\left(\widehat{n}+1\right)} \tag{15}
$$

we may finally write

$$
\begin{aligned}
\widehat{H}_2 \quad &\approx \nu\widehat{N} + \chi\left(\widehat{n}+\frac{1}{2}\right)\left(b+b^\dagger\right) + \lambda\sqrt{\widehat{n}+1}\sigma_z \\
&\quad + \frac{\chi^2}{\lambda\sqrt{\widehat{n}+1}}\left(b+b^\dagger\right)^2.
\end{aligned} \tag{16}
$$

This is a Hamiltonian that is already diagonal and direct to solve. The purpose of this contribution was to show that the Hamiltonian of the total interaction could be simplified. We have achieved this. A more complete study to look for the evolution of observables and possibilities of reconstructing the mirror-field interaction is still in preparation and will be published elsewhere.

4. Conclusions

We have shown that the problem of a quantized field interacting simultaneously with a two-level atom and a movable mirror may be diagonalized via a set of transformations, the main one, being a transformation that involves Susskind-Glogower operators, equation (6). This transformation, besides the fact that does not involve approximations, allows us to simplify the problem by "eliminating" the field operators to leave an effective interaction between atom and mirror. The Hamiltonian for this interaction then may be slightly rotated to obtain a dispersive Hamiltonian that being diagonal is already solvable such that the evolution operator may be eventually found.

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