Finite photonic lattices: a solution using characteristic polynomials

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We give an analytic solution for the propagation of light in a finite waveguide array when the interaction coefficients are constant. We show how to excite the different sites to allow propagation without perturbation.

Keywords: Light propagation; waveguide arrays.

Se presenta una solución analítica para la propagación de la luz en un arreglo finito de guías de onda cuando los coeficientes de la interacción son constantes. Mostramos como excitar los diferentes lugares para permitir una propagación sin perturbación.

Descriptores: Propagación de la luz; arreglos de guías de onda.

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1. Introduction

The discrete coupling or tunneling process between periodically arranged potential wells is a fundamental topic that has been extensively investigated in many branches of physics. In optics, arrays of weakly coupled waveguides and resonators are prime examples of such systems, where the coupling dynamics can be directly observed and investigated [1-5]. Periodic array structures are typically comprised form single-mode waveguides that are coupled to each other [5].

Makris and Christodoulides [6] considered a finite one-dimensional array of N waveguides, and by using a method of images, showed that the field at the nth site, provided the mth site is excited at $Z=0$, is given by

$$E_n(Z) = \sum_{r=-\infty}^{\infty} \{ i^{-(2N+2)r} [i^{n-m} J_{n-m-(2N+2)r}(2Z) - i^{n+m+2} J_{n+m+2-(2N+2)r}(2Z)] \}. \tag{1}$$

The purpose of the present contribution is to study finite photonic lattices and to show that by using its eigenmodes, which we will show are vectors with the characteristic polynomials as their elements, a periodic (Talbot-like [7]) or constant (unperturbed intensities during propagation) behavior may be produced. A byproduct of this contribution is that, by comparing our solutions to Eq. (1), it is possible to find an identity for such sums of Bessel functions.

2. The waveguide array

Let us consider a linear finite array of N weakly coupled waveguides. Within the context of coupled mode theory, wave propagation in such a structure is described by the following discrete linear Schrödinger-like equations [1-3]:

$$i\frac{dE_1}{dZ} + \omega_1 E_1 + \lambda E_2 = 0, \tag{2}$$

$$i\frac{dE_n}{dZ} + \omega_n E_n + \lambda\left(E_{n+1} + E_{n-1}\right) = 0, \quad n = 2, 3, ..., N-1, \tag{3}$$

and

$$i\frac{dE_N}{dZ} + \omega_N E_N + \lambda E_{N-1} = 0. \tag{4}$$

We can write in short form the above equations by using the matrix

$$A = \begin{pmatrix} \omega_1 & \lambda & 0 & 0 & ... & ... & 0 \\ \lambda & \omega_2 & \lambda & 0 & ... & ... & 0 \\ 0 & \lambda & \omega_3 & \lambda & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & \lambda & \omega_{N-1} & \lambda \\ 0 & ... & ... & ... & 0 & \lambda & \omega_N \end{pmatrix}, \tag{5}$$

so that we need to solve the differential equation

$$i\frac{d\vec{X}}{dZ} = A\vec{X}, \quad \vec{X}(Z=0) = \vec{X}(0), \tag{6}$$

where

$$\vec{X} = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ \vdots \\ E_N \end{pmatrix}. \tag{7}$$

In order to solve the system of equations, we need the eigenvalues of matrix A, which are obtained from the determinant relation

$$F_N(x) = 0 = \begin{vmatrix} \omega_1 - x & \lambda & 0 & 0 & 0 & 0 \\ \lambda & \omega_2 - x & \lambda & 0 & 0 & 0 \\ 0 & \lambda & \omega_3 - x & \lambda & 0 & 0 \\ \ldots & \ldots & \ldots & . & . & \ldots \\ 0 & \ldots & \ldots & \lambda & \omega_{N-1} - x & \lambda \\ 0 & \ldots & \ldots & 0 & \lambda & \omega_N - x \end{vmatrix}, \tag{8}$$

where the polynomial characteristic polynomial $F_N(x)$ can be computed by the three-term recurrence [8]

$$F_0(x) = 1, \quad F_1(x) = \frac{x-\omega_1}{\lambda},$$
$$F_n(x) = \frac{x-\omega_n}{\lambda} F_{n-1}(x) - F_{n-2}(x),$$
$$n = 2, 3, ..., N. \tag{9}$$

2.1. Propagation without perturbation

Once we have found the roots of the characteristic polynomial, *i.e.* the eigenvalues of the matrix A, we can find its eigenvectors, $A\vec{X}_j = x_j \vec{X}_j$, as

$$\vec{X}_j = \begin{pmatrix} F_0(x_j) \\ F_1(x_j) \\ \vdots \\ \vdots \\ F_{N-1}(x_j) \end{pmatrix}. \tag{10}$$

Therefore, if at $Z = 0$ we excite different sites of the waveguide array according to one of its eigenvectors, *i.e.* $E_n(Z=0) = F_{n-1}(x_j)$ with $n = 1, 2, 3, ..., N$, because $\exp(iZA)\vec{X}_j = \exp(iZx_j)\vec{X}_j$, the intensities at each site will remain constant throughout propagation.

The Fig. 1 below shows the propagation behavior for $N = 11$, $\omega_k = 0$ for $k = 1, 2, 3, ..., 6$ and $\lambda = 1$; the mode corresponds to the eigenvalue $x_1 = 2\cos(\pi/12)$.

The Fig. 2 shows the propagation dynamics when the first eigenmode ($x_1{=}2\cos(\pi/12)$) is launched into one finite waveguide array of 11 elements, $\omega_k{=}0$ for $k{=}1, 2, 3, ..., 11$ and $\lambda{=}1$.

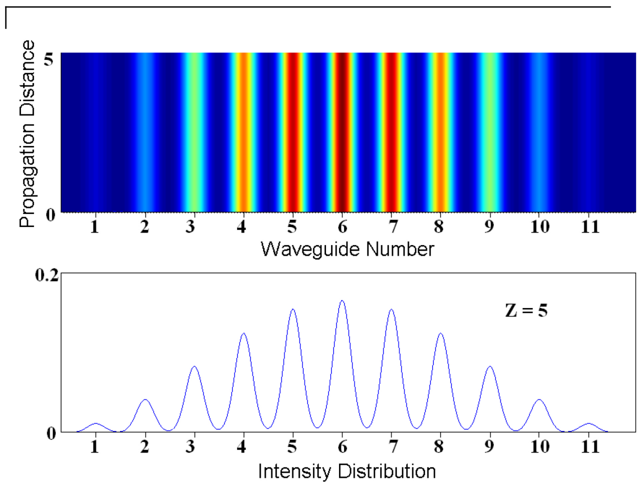

FIGURE 1. Propagation dynamics when the first eigenmode is launched into a finite waveguide array of 11 elements and (bottom) the corresponding intensity at $Z = 5$.

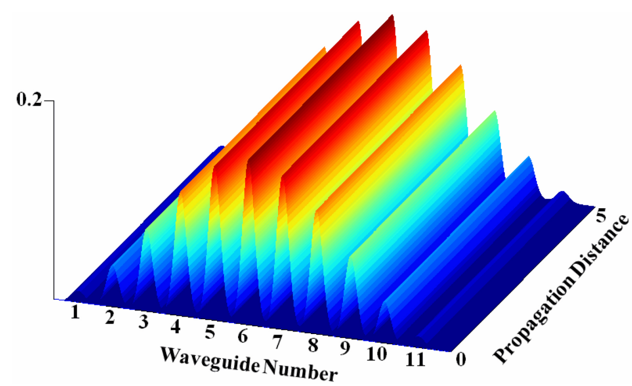

FIGURE 2. Side view of the invariant propagation of the first eigenmode, N=11.

3. Solution of the problem

The formal solution to the differential equation is then

$$\vec{X}(Z) = e^{iAZ}\vec{X}(0). \tag{11}$$

In order to calculate the matrix $\exp(iAZ)$, we write the A matrix as usual, $A = S\Lambda S^{-1}$, where Λ is the diagonal matrix of eigenvalues, and the base transformation matrix S is given by the matrix formed with the normalized eigenvectors,

$$S_{m,n} = \frac{F_m(x_n)}{\sqrt{\sum_{k=1}^{N} [F_{k-1}(x_n)]^2}} \quad (m, n = 1, 2, 3, ..., N)$$

We have to calculate the inverse S^{-1}. However, we can show that the S matrix is an orthogonal matrix, and then

$$S^{-1} = S^{\dagger} \tag{12}$$

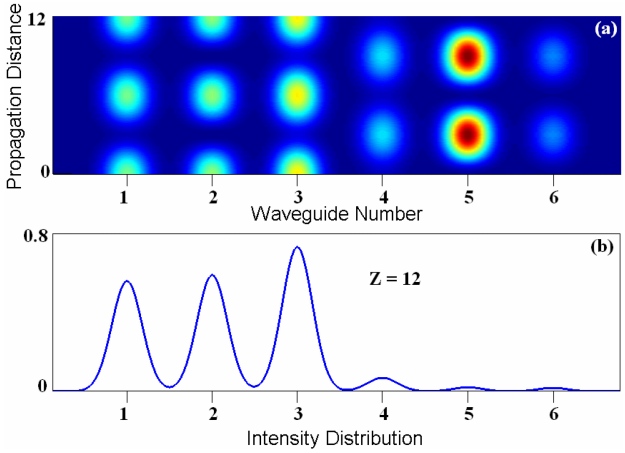

FIGURE 3. (a) Dynamical superposition of the second and third eigenmodes propagating into one array of 6 elements, and (bottom) its corresponding output intensity. Notice the periodicity in the "intensity carpet" (Talbot-like effect).

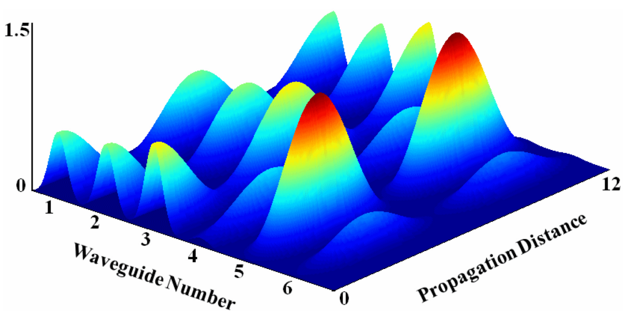

FIGURE 4. Side view of the intensity propagation of the superposition of the second and third eigenmodes, N=6.

or explicitly

$$\left(S^{-1}\right)_{m,n} = \frac{F_n\left(x_m\right)}{\sqrt{\sum_{k=1}^{N}\left[F_{k-1}\left(x_m\right)\right]^2}}$$

$$(m,n = 1,2,3,...,N) \, . \qquad (13)$$

Then the solution to the differential equation is

$$\vec{X}(Z) = e^{iAZ}\vec{X}(0) = S\exp\left(iZ\Lambda\right)S^{-1}\vec{X}\left(0\right) . \qquad (14)$$

We also present now graphically the propagation behavior when $N = 6$, $\omega_k = (-1)^k$ and $\lambda = 2$, when we have a superposition of the second ($x_2 \sim -3.740$) and third mode ($x_3 = -2.686$).

3.1. The Chebyshev polynomials case

If in the original matrix we take the special case when $\omega_k = 0$ for $k = 1, 2, 3, ..., N$ and $\lambda = 1$, the characteristic polynomial $F_N\left(x\right)$ reduces to the Chebyshev polynomials of the

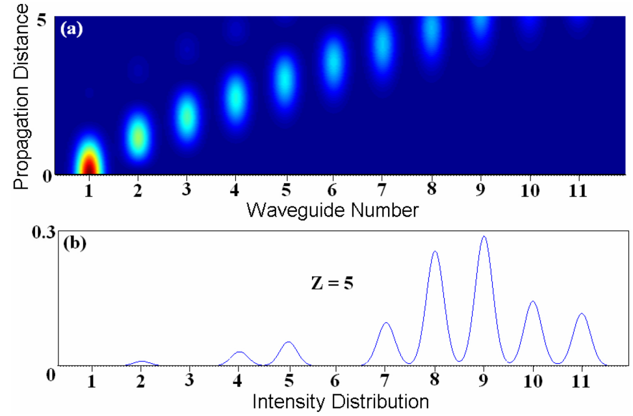

FIGURE 5. Intensity evolution when the first site is excited, N=11.

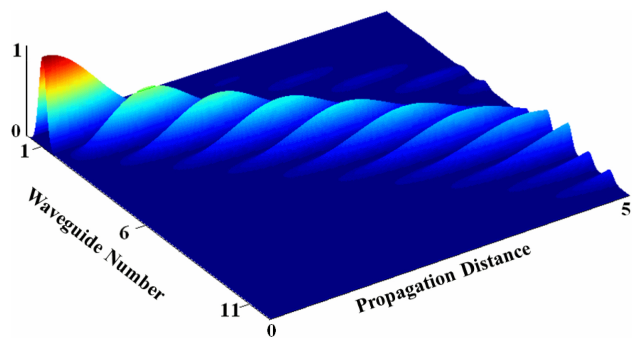

FIGURE 6. Side view of the intensity propagation when the first site is excited, N=6.

second kind, *i.e.*

$$F_N(x) \equiv U_N\left(\frac{x}{2}\right) = 0. \qquad (15)$$

and therefore, the N roots are given by

$$x_k = 2\cos\left(\frac{k\pi}{N+1}\right), \quad k = 1, ..., N. \qquad (16)$$

Similar expressions have been already derived by Yavir *et al* [9] and by Butler *et al* [10].

Furthermore, the diagonal matrix of eigenvalues is

$$\Lambda_{m,n} = 2\delta_{mn}\cos\left(\frac{m\pi}{N+1}\right)(m,n = 1,2,3,...,N), \quad (17)$$

and the base transformation matrix S is given by

$$S_{m,n} = \frac{U_m\left(x_n/2\right)}{\sqrt{\sum_{k=1}^{N}\left[U_{k-1}\left(x_n/2\right)\right]^2}}$$

$$(m,n = 1,2,3,...,N) \, , \qquad (18)$$

Then in this case we can write explicitly the solution (14),

$$X_n(Z) = \sum_{\beta=1}^{N}\sum_{\alpha=1}^{N} \frac{U_n(x_\alpha/2)\,U_{\beta-1}(x_\alpha/2)\exp\left[2iZ\cos\left(\frac{\alpha\pi}{N+1}\right)\right]X_\beta(0)}{\sqrt{\sum\limits_{k_1=1}^{N}\left[U_{k_1-1}(x_\alpha/2)\right]^2}\sqrt{\sum\limits_{k_2=1}^{N}\left[U_{k_2-1}(x_\alpha/2)\right]^2}} \quad (n = 1, 2, 3, ..., N)\,. \tag{19}$$

In the following figures we show the propagation behavior for this case (*i.e.* $\omega_k = 0$ for $k = 1, 2, 3, .., 11$ and $\lambda = 0$) when $N = 11$, and the first site is excited,

4. Connection with the images solution

In order to match our results with the ones of Makris and Christodoulides [6], we take the Chebyshev polynomials case ($\omega_k = 0$ for $k = 1, 2, 3, ..., N$ and $\lambda = 1$) and we excite the mth site of the finite array, putting $\left[\vec{X}(0)\right]_j = \delta_{jm}$. From (19), we get

$$E_n(Z) = \sum_{\alpha=1}^{N} \frac{U_n(x_\alpha/2)\,U_{m-1}(x_\alpha/2)\exp\left[2iZ\cos\left(\frac{\alpha\pi}{N+1}\right)\right]}{\sum\limits_{k=1}^{N}\left[U_{k-1}(x_\alpha/2)\right]^2} \tag{20}$$

deriving, as a byproduct, the following very complicated special functions relation between the Chebyshev polynomials of the second kind and the Bessel functions,

$$\sum_{\alpha=1}^{N} \frac{U_n\left[\cos\left(\frac{\alpha\pi}{N+1}\right)\right]U_{m-1}\left[\cos\left(\frac{\alpha\pi}{N+1}\right)\right]\exp\left[2iZ\cos\left(\frac{\alpha\pi}{N+1}\right)\right]}{\sum\limits_{k=1}^{N}\left\{U_{k-1}\left[\cos\left(\frac{\alpha\pi}{N+1}\right)\right]\right\}^2} =$$

$$= \sum_{r=-\infty}^{\infty}\{i^{-(2N+2)r}[i^{n-m}J_{n-m-(2N+2)r}(2Z) - i^{n+m+2}J_{n+m+2-(2N+2)r}(2Z)]\}. \tag{21}$$

5. Conclusions

We have solved the system of Eqs. (6) using the very particular property that the matrix formed by the normalized eigenvectors is orthogonal, giving directly the inverse of the matrix S. The particular matrix we use, *i.e.* the one that models the photonic lattice, was shown to produce "characteristic eigenvectors", that allowed either constant propagation, or, by using the superposition principle, Talbot-like effects [7].

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