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MODELING NON-LINEAR COHERENT STATES IN FIBER ARRAYS

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A class of nonlinear coherent states related to the Susskind-Glogower (phase) operators is obtained. We call these nonlinear coherent states as Bessel states because the coefficients that expand them into number states are Bessel functions. We give a closed form for the displacement operator that produces such states.

Keywords: Non-linear coherent states.

1. Introduction

Over the years, there has been major effort towards the generation of nonclassical states in different systems, such as electromagnetic fields, trapped ions, etc. Nonclassical states exhibit less fluctuations or noise than coherent states for certain observables. This is why coherent states noise is referred to as the standard quantum limit (SQL). Nonclassical states that have attracted the greatest interest include macroscopic quantum superpositions of quasiclassical coherent states,[1,2] squeezed states,[3] whose fluctuations in one of the quadratures or the amplitude are reduced the SQL and the particularly important limit of extreme amplitude squeezing, namely, Fock states.[4] One may think of several systems that may generalize the harmonic oscillator in order to produce such NCS, for instance, we may consider time dependent frequencies,[5] or a type that has recently attracted great interest is to deform the harmonic oscillator to generate so-called non-linear coherent states,[6,7] that may be related to q-deformed algebras.[a]

A q-deformed algebra was used to introduce the idea of quantum q-oscillators, whose interpretation[8,9] was as a nonlinear oscillator with a very specific type of nonlinearity,

[a] q-deformed algebras are deformed versions of the standard Lie algebras, which are recovered as the deformation parameter q goes to unity. The basic interest in q-deformed algebras resides in the fact that they encompass a set of symmetries that is richer than that of the standard Lie algebras. q-deformed algebras could be a useful tool to describe physical system symmetries that cannot be properly treated within Lie algebras.

in which the frequency of vibration depends on the energy of these vibrations through the hyperbolic cosine function containing a nonlinear parameter. This observation suggested that there would exist other types of nonlinearities for which the frequency of oscillation varies with the amplitude in a different manner from the one obtained with the q-deformed algebra. Such oscillators are called f-oscillators.[8] One can extend the notion of coherent states by using f-oscillators to construct f-coherent states (also called nonlinear coherent states) by means of "deformed" creation and annihilation operators representing the dynamical variables to be associated with the quantum f-oscillators.[7] These operators are defined through

$$A = af(N) = f(N+1)a, \quad A^\dagger = f(N)a^\dagger = a^\dagger f(N+1), \tag{1}$$

with a and $a^\dagger$ the annihilation and creation operators for the harmonic oscillator and $N = a^\dagger a$ is the number operator.

The importance of studying nonlinear coherent states resides in their physical consequences such as amplitude squeezing, quantum interferences and the possibility of having super- or sub-Poissonian statistics. Furthermore, nonlinear coherent states may be realized in the motion of a trapped ion.[6,10]

The modelling of quantum mechanical systems with classical optics is a topic that has attracted interest recently. Along these lines Man'ko *et al.* have proposed to realize quantum computation by quantum like systems[11] and Crasser *et al.*[12] have pointed out the similarities between quantum mechanics and Fresnel optics in phase space. Following these cross-applications, here we would like to show how a nonlinear coherent state may be modelled in a fiber array.[13] Therefore, the purpose of the present work is twofold: to show how to use quantum optics methods to solve classical optics propagation problems and create a classical system to emulate a quantum one showing the potential for studying quantum optics with classical systems.

2. Susskind-Glogower Operators

The annihilation and creation Susskind-Glogower[14] operators may be defined as

$$V = \frac{1}{\sqrt{aa^\dagger}}a, \quad V^\dagger = a^\dagger\frac{1}{\sqrt{aa^\dagger}}, \tag{2}$$

i.e. as the definitions of deformed creation and annihilation operators given in (1) and (2). We can verify that $VV^\dagger = 1$ but $V^\dagger V = 1 - |0\rangle\langle 0|$, that gives the commutation relation $[V, V^\dagger] = |0\rangle\langle 0|$, that makes it complicated to calculate the exponential (displacement operator by analogy to normal annihilation and creation operators)

$$\mathcal{D}(\alpha) = e^{\alpha V^\dagger - \alpha^* V}. \tag{3}$$

The commutation relation for the Susskind-Glogower operators do not allow the application of the Baker-Hausdorff formula[15] or even to propose an *ansatz* that would work properly for the factorization of (3) in the products of exponentials. Instead we can try to develop the exponential (3) in a Taylor series, and then to evaluate the terms $(V+V^\dagger)^k$. For instance, for $k=7$ we have

$$\begin{aligned}(V+V^\dagger)^7 = &\{(V+V^\dagger)^7\}_A - \binom{7}{2}(|1\rangle\langle 0| + |0\rangle\langle 1|)\\ &- \binom{7}{1}(|3\rangle\langle 0| + |0\rangle\langle 3| + |2\rangle\langle 1| + |1\rangle\langle 2|)\\ &- \binom{7}{0}(|5\rangle\langle 0| + |0\rangle\langle 5| + |4\rangle\langle 1| + |1\rangle\langle 4| + |2\rangle\langle 3| + |3\rangle\langle 2|) \end{aligned} \tag{4}$$

where $\{\}_A$ means antinormal order, that is, to arrange terms such that the powers of the operator V are always to the left of powers of the operator $V^\dagger$. Note that in the above equation, the term multiplying $\binom{7}{2}$ are all the possible combinations for one phonon (photon in the case of the quantized electromagnetic field), the term multiplying $\binom{7}{1}$ are all the combinations for three phonons and the term multiplying $\binom{7}{0}$ all the combinations for five phonons.

3. Coherent States from Application of Displacement Operator to the Vacuum

We define coherent states as

$$|\alpha\rangle_{SG} = \mathcal{D}(\alpha)|0\rangle. \tag{5}$$

From (5) we can write[b]

$$|ix\rangle_{SG} = e^{ix(V+V^\dagger)}|0\rangle = e^{ixV}e^{ixV^\dagger}|0\rangle - \sum_{k=0}^{\infty}\frac{(ix)^k}{k!}\sum_{n=0}^{[\frac{k}{2}-1]}\binom{k}{n}|k-2n-2\rangle \tag{6}$$

where $[(k/2)-1]$ is the floor function, also called the greatest integer function or integer value, gives the largest integer less than or equal to $(k/2)-1$. We can rewrite the above equation as

$$|ix\rangle_{SG} = e^{ixV}e^{ixV^\dagger}|0\rangle - V^2\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}\sum_{n=0}^{[\frac{k}{2}-1]}\binom{k}{n}V^{2n}V^{\dagger k}|0\rangle, \tag{7}$$

and take the second sum to ∞ as we would add only zeros

$$|ix\rangle_{SG} = e^{ixV}e^{ixV^\dagger}|0\rangle - V^2\sum_{k=0}^{\infty}\frac{(ix)^k}{k!}\sum_{n=0}^{\infty}\binom{k}{n}V^{2n}V^{\dagger k}|0\rangle. \tag{8}$$

[b]For simplicity we use $\alpha = ix$, however it may be easily generalized to a complex number by a using a transformation of the form $e^{i\theta a^\dagger a}$. This is: $e^{i\theta a^\dagger a}|ix\rangle_{SG} = e^{i\theta a^\dagger a}e^{ix(V+V^\dagger)}|0\rangle = |ixe^{i\theta}\rangle_{SG}$.

We now exchange the order of the sums

$$|ix\rangle_{SG} = e^{ixV}e^{ixV^\dagger}|0\rangle - V^2\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(ix)^k}{(k-n)!n!}V^{2n}V^{\dagger k}|0\rangle. \tag{9}$$

By taking $m = k - n$ we finally write

$$|ix\rangle_{SG} = e^{ixV}e^{ixV^\dagger}|0\rangle - V^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(ix)^{(m+n)}}{m!n!}V^{2n}V^{\dagger m+n}|0\rangle, \tag{10}$$

or

$$|ix\rangle_{SG} = (1 - V^2)e^{ixV}e^{ixV^\dagger}|0\rangle. \tag{11}$$

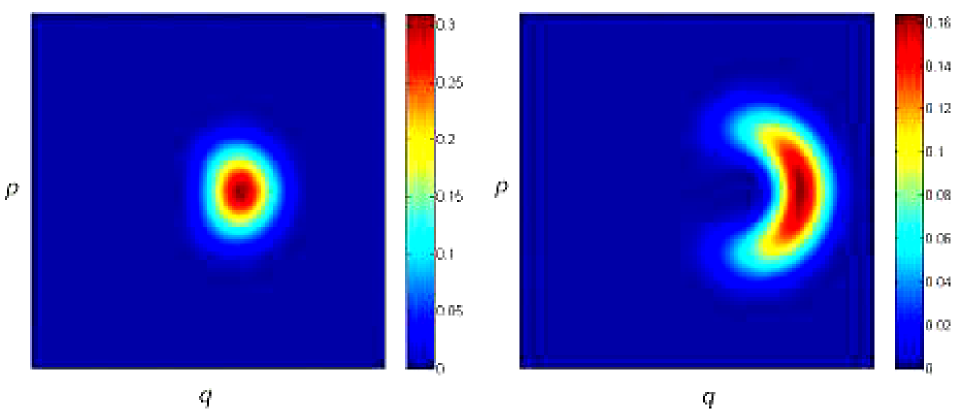
3.1. *Bessel states*

Application of the nonlinear displacement operator to the vacuum then gives

$$|ix\rangle_{SG} = \mathcal{D}(ix)|0\rangle = -\frac{i}{x}\sum_{n=0}^{\infty}(n+1)i^n J_{n+1}(2x)|n\rangle. \tag{12}$$

In Fig. 1 we plot the Q function[c] for several amplitudes. We can see banana shaped states that are typical of some other nonlinear systems such as Kerr medium.[16] It is also possible to see that for large values of the amplitude a superposition of two distinguishable states arises. This states will show squeezing in the amplitude.[4] This may be clearly seen by plotting the Mandel-$\mathcal{Q}$ parameter[17]

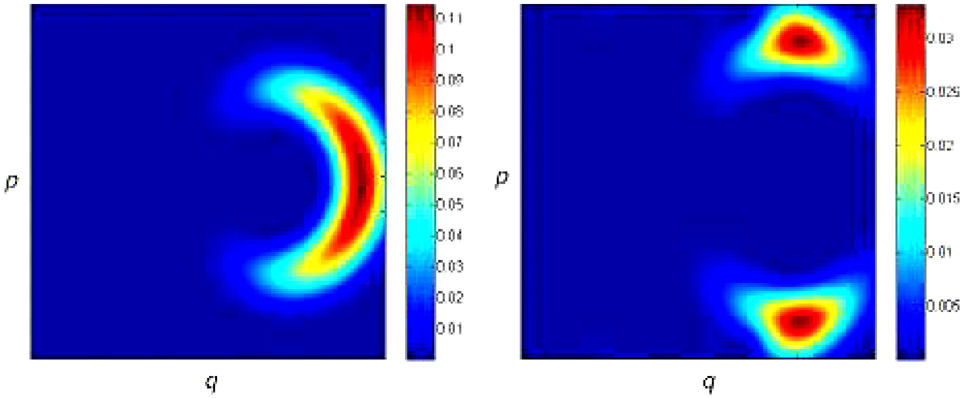
$$\mathcal{Q} = \frac{\langle(a^\dagger a)^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} - 1 \tag{13}$$



(a) (b)

Fig. 1. Q-function for the rotated Bessel state, $|x\rangle_{SG} = e^{-iN\pi/2}|ix\rangle_{SG}$, with (a) $x = 1$, (b) $x = 5$, (c) $x = 10$, and (d) $x = 20$.

[c] $Q(\beta = |\langle\beta|\alpha\rangle_{SG}|^2/\pi$ with $|\beta\rangle$ a coherent state.



(c) (d)

Fig. 1. (*Continued*)

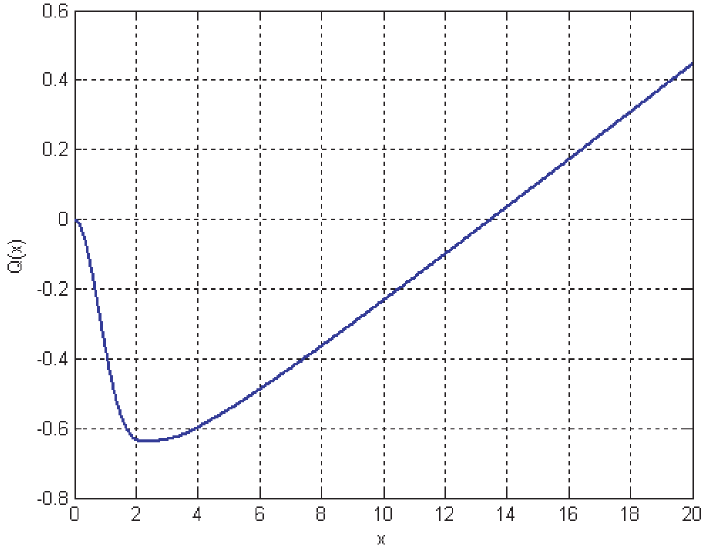


Fig. 2. Mandel-$\mathcal{Q}$ parameter for the Bessel state as a function of the amplitude x.

which, if $\mathcal{Q} < 0$ shows sub-Poissonian features. In Fig. 2 we show a plot this parameter, and we can see that the state is sub-Poissonian from zero to large amplitudes ($\approx$12).

4. Non-Linear Displaced Number States and Fiber Arrays

The analogy to a classical optical system comes from noticing that the system we are solving is similar to the propagation of light through a semi-infinite fiber array.[13]

In such a system, the differential equations to solve are

$$i\frac{dE_n}{dz} + c(E_{n+1} + E_{n-1}) = 0, \quad n \geq 1, \tag{14}$$

and

$$i\frac{dE_0}{dz} + cE_1 = 0, \tag{15}$$

where E_n is the electric modal field in the nth waveguide of the fiber array.

If we consider a non-linear Hamiltonian of the form[d]

$$H = \eta(V + V^\dagger) \tag{16}$$

with η proportional to the Lamb-Dicke parameter[18] in the trapped ion-laser interaction case. We can solve the Schrödinger equation

$$i\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle \tag{17}$$

for this Hamiltonian by expanding the wave function into number states

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} E_n(t)|n\rangle, \tag{18}$$

where $E_n(t)$ are the coefficients of the expansion. By plugging the wave function above into the Schrödinger equation, we obtain a system of differential equations for the coefficients $E_n(t)$ which is in fact the system of Eqs. (14) and (15) with $cz = -\eta t$. Therefore we can borrow the solution by Makris and Christodoulides[13]

$$E_n(t) = i^{n-m}J_{n-m}(-2\eta t) + i^{n+m}J_{n+-m+2}(-2\eta t), \tag{19}$$

for an initial condition $|\psi(0)\rangle = |m\rangle$. Therefore the wave function

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}(i^{n-m}J_{n-m}(-2\eta t) + i^{n+m}J_{n+m+2}(-2\eta t))|n\rangle, \tag{20}$$

corresponds to a non-linear displaced number state, i.e. application of the non-linear displacement operator to a number state.

5. Conclusion

We have shown that new quantum states, namely Bessel states, may be generated, for instance, in ion traps by properly engineering a nonlinear Hamiltonian. In this case it corresponds to a Hamiltonian given by the sum of the Susskind-Glogower phase operators. We have not considered the effect of dissipation, but it has been

[d] Nonlinear interactions may be properly engineered in ion traps[18] taking advantage of the Laguerre polynomials properties.[19]

shown already that the wave function of an ion or an electromagnetic field may be reconstructed even though dissipation occurs.[20]

Finally, it is worth noting that, although we cannot apply Baker-Hausdorff formula to write the nonlinear displacement operator as a product of exponentials, it is possible to find the evolution operator, $\mathcal{D}(-i\eta t)$, for the Hamiltonian (16): from (20) we have

$$\langle k|\psi(t)\rangle = i^{k-m}J_{k-m}(-2\eta t) + i^{k+m}J_{k+m+2}(-2\eta t) = \langle k|e^{-i\eta t(V+V^{\dagger})}|m\rangle, \tag{21}$$

from which the evolution operator may be written as

$$\mathcal{D}(-i\eta t) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(i^{k-m}J_{k-m}(-2\eta t) + i^{k+m}J_{k+m+2}(-2\eta t)\right)|k\rangle\langle m|. \tag{22}$$

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